Robust multi-port cascade calculations using decomposed matrix parameters

Rob F.M. van den Brink

Abstract—This paper defines a solid framework for performing multi-port calculations with full matrix parameters in a generic, systematic and elegant way. It enhances this approach with a concept that we have formalized with the name "decomposed" matrix parameters to simplify calculations with cascadable multi-ports. It discusses the consequences of using straight forward matrix multiplications with chain parameters for calculating a cascade of multi-ports. Especially when such an approach is applied to cables with high insertion loss, which will easily occur with twisted-pair telephony cables used for broadband deployments, the calculated result can easily suffer from significant round-off errors. This paper demonstrates this problem and introduces an alternative sets of algorithms for cascade calculations, that are directly expressed in S-, Z- or Y-parameters. The proposed calculation framework offers a good basis for implementing generic toolboxes for multi-port analyses. It treats waves, voltages and currents in an equal manner, and keeps its elegance for an arbitrary ordering of port numbers.

Index Terms—multi port matrix methods, s-parameters, chain parameters, cascade calculations, twisted pair cables, transmission line theory, cable modeling.

1 Introduction

The subject of multi port calculations on multi wire pair cables arises in all kinds of studies related to broadband deployments via existing telephony cabling [1], [2]. Recent developments on high speed DSL modems (Digital Subscriber Line) enabled the delivery of bitrates up to 1 Gb/s over these twisted pair cables, using frequencies that are far beyond the design targets of these cables. Recent versions of VDSL transmits signals up to 35 MHz, G.fast transmits up to 106MHz, and future version of G.fast are expected to transmit up to 212 MHz through these cables [3], [4].

The use of such high frequencies stimulated the development of advanced two-port cable models up to hundreds of MHz [5], [6], including advanced multi-port measurements [7], [8], [9] on these cables to analyse cross talk over wide frequency bands. To understand second-order effects in this crosstalk, we analyzed the so called "dual slope" effect in EL-FEXT [10], modelled it with multi-port matrix methods and calculated the overall transfer through a cascade with thousands of short cable sections. The numerical instability problems with well-known cascade calculations based on chain parameters required the development of more robust algorithms.

It is a common approach to use all kinds of matrix parameters for representing multi-ports [11], [12], [13], [14], [15], [16], for solving specific problems expressed in waves or other problems expressed in voltages and currents. But it is not trivial how to bring well-known two-port concepts into a generic multi-port software toolbox, while keeping it all very systematic and elegant for an arbitrary ordering of port numbers. Moreover, if such a toolbox implements multi-port cascade calculations via straight forward matrix multiplications with chain parameters, then it will suffer from severe round-off errors when applied to cables with high insertion losses.

This paper proposes a robust and generic set of algorithms for all kinds of multi-port cascade calculations, that does not suffer from such round-off errors. It is defined on top of a solid framework with matrix parameters. Chapter 2 starts with defining a systematic and elegant framework for multi-ports with "full" matrix parameters, and shows how to convert various parameters in each other. Chapter 3 extends that framework for cascadable multi-ports by adding a concept that we have formalized with the name "decomposed" matrix parameters. Chapter 4 discusses the concept of straight-forward cascade calculations via chain parameters and shows that such an approach will suffer from round-off errors when applied to cables with very high insertion losses. It proposes alternative algorithms that are more robust and directly expressed in decomposed S-, Z- or Y-parameters. Chapter 5 extends the framework for special classes of cascadable multi-ports that are reciprocal, symmetrical or both.

2 Full Matrix Parameters

It is a well-known concept of specifying linear devices in terms of matrix parameters. The use of two-port Z and Y parameters were introduces somewhere in the twenties, and two-port S-parameters were formalized in the fifties [17]. The two-port approach can be generalized into a similar multi-port approach, and this chapter offers a compact definition framework and associated terminology as a basis for the decomposition concept in chapter 3.

Rob F.M. van den Brink is with TNO, Den Haag, The Netherlands, e-mail: Rob.vandenBrink@tno.nl.

2.1 MULTI-PORT S, Z AND Y PARAMETERS

The signals at the ports of a device can be described as a pair of waves (W_a, W_b) or as a combination of voltage and current (U,I). It is a matter of preference which of these two are preferred, since both can be interchanged. By definition, an incident wave W_a flows into the multi-port, and a reflected wave W_b flows away from the multi-port. Similarly, currents (I) are by definition always directed into the multi-port and voltages are always between the port terminal (positive) and a common terminal (negative).

If the waves traveling through a device port are both normalized to a (real) reference impedance R_n, then a wave pairs can easily be converted into a voltage/current pair (or reverse) as expressed in (1).

$$\begin{array}{rcl} U & = & (W_a + W_b) \cdot \sqrt{R_n} \\ I & = & (W_a - W_b)/\sqrt{R_n} \\ & & and \\ W_a & = & (U + R_n \cdot I)/(2 \cdot \sqrt{R_n}) \\ W_b & = & (U - R_n \cdot I)/(2 \cdot \sqrt{R_n}) \end{array} \quad (1)$$

If the device is linear, then these signals are all related via linear equations. If that linear device has N ports then we need N of these equations for a full multi-port description with an $N \times N$ matrix, but there are many ways to organize them. Relating all waves with each other can be expressed in a compact matrix format as $\mathbf{W}_b = \mathbf{S} \times \mathbf{W}_a$ or in an expanded matrix format as:

$$\begin{bmatrix} W_{b1} \\ W_{b2} \\ \cdots \\ W_{bN} \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1N} \\ s_{21} & s_{22} & \cdots & s_{2N} \\ \cdots & \cdots & \cdots & \cdots \\ s_{N1} & s_{N2} & \cdots & s_{NN} \end{bmatrix} \times \begin{bmatrix} W_{a1} \\ W_{a2} \\ \cdots \\ W_{aN} \end{bmatrix} \quad (2)$$

Similarly we can do the same for voltages and currents, resulting in similar matrix relations like $\mathbf{U} = \mathbf{Z} \times \mathbf{I}$ and $\mathbf{I} = \mathbf{Y} \times \mathbf{U}$. The matrices $\mathbf{S}$, $\mathbf{Z}$, $\mathbf{Y}$ are subsequently called the (full) scatter-, impedance- and admittance-matrices. And since the definitions of their signals are related, these matrices $\mathbf{S}$, $\mathbf{Z}$, $\mathbf{Y}$ can be converted in each other as summarized in (3). Matrix $\mathbb{I}$ refers to the identity matrix with all ones on its diagonal and zeros elsewhere.

$$\begin{array}{rcl} \mathbf{S} & = & inv(\mathbf{Z}/R_n + \mathbb{I}) \times (\mathbf{Z}/R_n - \mathbb{I}) \\ \mathbf{S} & = & inv(\mathbb{I} + \mathbf{Y} \cdot R_n) \times (\mathbb{I} - \mathbf{Y} \cdot R_n) \\ \mathbf{Z} & = & inv(\mathbb{I} - \mathbf{S}) \times (\mathbb{I} + \mathbf{S}) \cdot R_n \\ \mathbf{Z} & = & inv(\mathbf{Y}) \\ \mathbf{Y} & = & inv(\mathbb{I} + \mathbf{S}) \times (\mathbb{I} - \mathbf{S})/R_n \\ \mathbf{Y} & = & inv(\mathbf{Z}) \end{array} \quad (3)$$

The computational advantage of using waves and S-parameters instead of Z or Y-parameters is that infinite values do not occur for common real world networks. Examples are ideal shorts (zero in impedance but infinite in admittance) or ideal opens (zero in admittance but infinite in impedance).

2.2 Multi-port chain parameters (T, A)

Chain parameters result from yet another way of ordering the linear equations into a matrix format. They come in view as an alternative description for multi-ports where half of the ports are labeled as "input" and the other half as "output". Once this labeling has been defined for a particular multi-port application, the associated chain parameters will define the relation between input and output signals for that particular labeling definition. There are different conventions in use on how to order the input and output signals in such matrix expressions, but in this paper we use the convention that all input signals come first and all output signals come last. For instance, when we group all port numbers related to the input into an index vector, for instance $inp = [1, 3, 7, 2]$, and we do the same for the output, like in $out = [5, 4, 8, 6]$, then $\mathbf{W}_{ai} = \mathbf{W}_a(inp)$ refers to a sub vector of $\mathbf{W}_a$ contains only those wave values associated with the port numbers specified by inp, and $\mathbf{W}_{ao} = \mathbf{W}_a(out)$ to those associated with the output. By using this convention, we define the forward chain matrices $\mathbf{T}$ and $\mathbf{A}$ as in (4).

$$\begin{array}{c} \begin{bmatrix} \mathbf{W}_a(inp) \\ \mathbf{W}_b(inp) \end{bmatrix} = \mathbf{T} \times \begin{bmatrix} \mathbf{W}_b(out) \\ \mathbf{W}_a(out) \end{bmatrix} \\ and \\ \begin{bmatrix} \mathbf{U}(inp) \\ \mathbf{I}(inp) \end{bmatrix} = \mathbf{A} \times \begin{bmatrix} +\mathbf{U}(out) \\ -\mathbf{I}(out) \end{bmatrix} \end{array} \quad (4)$$

The minus sign for the vector with output currents is to create a relation between input and output where all currents are directed in forward direction. In the special case that matrix $\mathbf{A}$ represents a two-port then its (four) matrix parameters are also known as "ABCD-parameters".

3 DECOMPOSED MATRIX PARAMETERS

It is possible to specify relations between multi-port parameters by using their full matrix form, like transforming one of {$\mathbf{S}$, $\mathbf{Z}$, $\mathbf{Y}$} into one of {$\mathbf{T}$, $\mathbf{A}$} or back. But then some expressions will not become as simple as desired. Therefore a more convenient alternative is to "decompose" each of these full matrices into four decomposed sub-matrices, and to express the relations only between these decomposed matrices. It means that a set of four decomposed matrices is to be considered as one entity for describing a multi-port as a whole. This is explained below in more detail.

3.1 DECOMPOSING MATRIX S, Z AND Y

To perform calculations on a cascade of multi-ports in a convenient manner with $\mathbf{S}$, $\mathbf{Z}$ or $\mathbf{Y}$, our first step is to decouple the (arbitrary) port numbering from the calculation. This can be achieved by breaking the full matrix into four smaller parts, each of them relating signal transfer functions between input and/or output ports. To specify that concept, we will use the Matlab matrix syntax based on the following convention: if x and y are row vectors with port indices, then matrix $\mathbf{S}(x, y)$ denotes a sub-matrix of $\mathbf{S}$ having only its elements at the crossings of all rows denoted by x and all columns denoted by y. With this convention, we define the four decomposed sub-matrices {$\mathbf{S}_{ii}$, $\mathbf{S}_{oi}$, $\mathbf{S}_{io}$, $\mathbf{S}_{oo}$} from $\mathbf{S}$ as specified in (5). The equations in (6) refer to the same concept, but phrased in a different manner. Similarly we can define the four decomposed sub-matrices {$\mathbf{Z}_{ii}, \mathbf{Z}_{oi}, \mathbf{Z}_{io}, \mathbf{Z}_{oo}$ } from $\mathbf{Z}$ and {$\mathbf{Y}_{ii}$, $\mathbf{Y}_{oi}$, $\mathbf{Y}_{io}$, $\mathbf{Y}_{oo}$} from $\mathbf{Y}$ via similar expressions.

$$\begin{array}{lcll} \mathbf{S}_{ii} & = & \mathbf{S}(inp, inp) & = \text{input reflection and NEXT} \\ \mathbf{S}_{oi} & = & \mathbf{S}(out, inp) & = \text{forward transmission and FEXT} \\ \mathbf{S}_{io} & = & \mathbf{S}(inp, out) & = \text{reverse transmission and FEXT} \\ \mathbf{S}_{oo} & = & \mathbf{S}(out, out) & = \text{output reflection and NEXT} \end{array} \quad (5)$$

$$\begin{array}{c} \mathbf{W}_b(inp) = \mathbf{S}_{ii} \times \mathbf{W}_a(inp) + \mathbf{S}_{io} \times \mathbf{W}_a(out) \\ \mathbf{W}_b(out) = \mathbf{S}_{oi} \times \mathbf{W}_a(inp) + \mathbf{S}_{oo} \times \mathbf{W}_a(out) \\ or \\ \mathbf{W}_{bi} = \mathbf{S}_{ii} \times \mathbf{W}_{ai} + \mathbf{S}_{io} \times \mathbf{W}_{ao} \\ \mathbf{W}_{bo} = \mathbf{S}_{oi} \times \mathbf{W}_{ai} + \mathbf{S}_{oo} \times \mathbf{W}_{ao} \end{array} \quad (6)$$

A first observation of decomposing $\mathbf{S}$ in this way is that the decomposed sub-matrices have clear physical meaning in combination with multi-pair cabling. Matrix $\mathbf{S}_{oi}$ contains all transfer functions in forward direction, and represents all direct transmissions and far-end crosstalk values (FEXT) through that cable. The same applies for $\mathbf{S}_{io}$ in reverse direction. Matrix $\mathbf{S}_{ii}$ contains all transfer functions at the input, representing all reflections and near-end crosstalk values (NEXT) at one side of the cable. The same applies for $\mathbf{S}_{oo}$ at the other side. And if the multi port is to be used in reverse direction, then simply swap $\mathbf{S}_{oi}$ with $\mathbf{S}_{io}$ and swap $\mathbf{S}_{ii}$ with $\mathbf{S}_{oo}$.

3.2 DECOMPOSING MATRIX T AND A

The decomposition of matrices $\mathbf{T}$ and $\mathbf{A}$ are defined in a slightly different manner since the ordering of port indices has already taken into account when $\mathbf{T}$ and $\mathbf{A}$ were defined as a whole. They relate, by definition, the input signals with the output signals as expressed in (7) and (8), while (9) and (10) do the same but phrased in a different manner.

$$\begin{bmatrix} \mathbf{W}_a(inp) \\ \mathbf{W}_b(inp) \end{bmatrix} = \begin{bmatrix} \mathbf{T}_f & \mathbf{T}_a \\ \mathbf{T}_b & \mathbf{T}_r \end{bmatrix} \times \begin{bmatrix} \mathbf{W}_b(out) \\ \mathbf{W}_a(out) \end{bmatrix} \quad (7)$$

$$\begin{bmatrix} \mathbf{U}(inp) \\ \mathbf{I}(inp) \end{bmatrix} = \begin{bmatrix} \mathbf{A}_v & \mathbf{A}_z \\ \mathbf{A}_y & \mathbf{A}_c \end{bmatrix} \times \begin{bmatrix} +\mathbf{U}(out) \\ -\mathbf{I}(out) \end{bmatrix} \quad (8)$$

$$\begin{array}{c} \mathbf{W}_{ai} = \mathbf{T}_f \times \mathbf{W}_{bo} + \mathbf{T}_a \times \mathbf{W}_{ao} \\ \mathbf{W}_{bi} = \mathbf{T}_b \times \mathbf{W}_{bo} + \mathbf{T}_r \times \mathbf{W}_{ao} \end{array} \quad (9)$$

$$\begin{array}{c} \mathbf{U}_i = \mathbf{A}_v \times \mathbf{U}_o - \mathbf{A}_z \times \mathbf{I}_o \\ \mathbf{I}_i = \mathbf{A}_y \times \mathbf{U}_o - \mathbf{A}_c \times \mathbf{I}_o \end{array} \quad (10)$$

The indices in the sub matrices $\{\mathbf{T}_f, \mathbf{T}_a, \mathbf{T}_b, \mathbf{T}_r\}$ refer to forward-, incident-, reflected- and reverse T parameters respectively, and the indices in $\{\mathbf{A}_v, \mathbf{A}_y, \mathbf{A}_z, \mathbf{A}_c\}$ to voltage-, admittance-, impedance- and current A parameters respectively. In the special case that $\mathbf{T}$ represents an eight-port, then $\mathbf{T}_{ii} = \mathbf{T}(1:4, 1:4)$ and $\mathbf{T}_{oi} = \mathbf{T}(5:8, 1:4)$, where $5:8$ refers to a row vector with the indices $[5, 6, 7, 8]$ and $1:4$ to $[1, 2, 3, 4]$.

3.3 RELATION BETWEEN {T,A} AND {S,Z,Y}

An advantage of using decomposed multi-port matrices is the simplicity of expressing transformations between the decomposed version of {$\mathbf{T}$,$\mathbf{A}$} and {$\mathbf{S}$,$\mathbf{Z}$,$\mathbf{Y}$}. Table 1 summarizes a variety of such transformations, all phrased in a manner and order that minimizes the number of inverse matrix calculations and reuses intermediate results as much as possible. The waves at the ports are assumed to be all normalized to the same reference impedance R_n.

TABLE 1
Transformation equations between the four decomposition matrices between {T,A} and {S,Z,Y} parameters.

$\mathbf{T}_f = inv(\mathbf{S}_{oi})$	1a	$\mathbf{S}_{oi} = inv(\mathbf{T}_f)$	2a
$\mathbf{T}_a = -\mathbf{T}_f \cdot \mathbf{S}_{oo}$	1b	$\mathbf{S}_{oo} = -\mathbf{S}_{oi} \cdot \mathbf{T}_a$	2b
$\mathbf{T}_b = \mathbf{S}_{ii} \cdot \mathbf{T}_f$	1c	$\mathbf{S}_{ii} = \mathbf{T}_b \cdot \mathbf{S}_{oi}$	2c
$\mathbf{T}_r = \mathbf{S}_{io} - \mathbf{T}_b \cdot \mathbf{S}_{oo}$	1d	$\mathbf{S}_{io} = \mathbf{T}_r - \mathbf{S}_{ii} \cdot \mathbf{T}_a$	2d
$\mathbf{A}_y = inv(\mathbf{Z}_{oi})$	3a	$\mathbf{Z}_{oi} = inv(\mathbf{A}_y)$	4a
$\mathbf{A}_c = \mathbf{A}_y \cdot \mathbf{Z}_{oo}$	3b	$\mathbf{Z}_{oo} = \mathbf{Z}_{oi} \cdot \mathbf{A}_c$	4b
$\mathbf{A}_v = \mathbf{Z}_{ii} \cdot \mathbf{A}_y$	3c	$\mathbf{Z}_{ii} = \mathbf{A}_v \cdot \mathbf{Z}_{oi}$	4c
$\mathbf{A}_z = \mathbf{A}_v \cdot \mathbf{Z}_{oo} - \mathbf{Z}_{io}$	3d	$\mathbf{Z}_{io} = \mathbf{A}_v \cdot \mathbf{Z}_{oo} - \mathbf{A}_z$	4d
$\mathbf{A}_z = -inv(\mathbf{Y}_{oi})$	5a	$\mathbf{Y}_{oi} = -inv(\mathbf{A}_z)$	6a
$\mathbf{A}_v = \mathbf{A}_z \cdot \mathbf{Y}_{oo}$	5b	$\mathbf{Y}_{oo} = -\mathbf{Y}_{oi} \cdot \mathbf{A}_v$	6b
$\mathbf{A}_c = \mathbf{Y}_{ii} \cdot \mathbf{A}_z$	5c	$\mathbf{Y}_{ii} = -\mathbf{A}_c \cdot \mathbf{Y}_{oi}$	6c
$\mathbf{A}_y = \mathbf{Y}_{io} + \mathbf{A}_c \cdot \mathbf{Y}_{oo}$	5d	$\mathbf{Y}_{io} = \mathbf{A}_y - \mathbf{A}_c \cdot \mathbf{Y}_{oo}$	6d
$\mathbf{A}_v = (\mathbf{T}_f + \mathbf{T}_b + \mathbf{T}_a + \mathbf{T}_r)/2$			7a
$\mathbf{A}_c = (\mathbf{T}_f - \mathbf{T}_b - \mathbf{T}_a + \mathbf{T}_r)/2$			7b
$\mathbf{A}_y = (\mathbf{T}_f - \mathbf{T}_b + \mathbf{T}_a - \mathbf{T}_r)/Rn/2$			7c
$\mathbf{A}_z = (\mathbf{T}_f + \mathbf{T}_b - \mathbf{T}_a - \mathbf{T}_r) \cdot Rn/2$			7d
$\mathbf{T}_f = (\mathbf{A}_v + \mathbf{A}_y \cdot R_n + \mathbf{A}_z/R_n + \mathbf{A}_c)/2$			8a
$\mathbf{T}_r = (\mathbf{A}_v - \mathbf{A}_y \cdot R_n - \mathbf{A}_z/R_n + \mathbf{A}_c)/2$			8b
$\mathbf{T}_b = (\mathbf{A}_v - \mathbf{A}_y \cdot R_n + \mathbf{A}_z/R_n - \mathbf{A}_c)/2$			8c
$\mathbf{T}_a = (\mathbf{A}_v + \mathbf{A}_y \cdot R_n - \mathbf{A}_z/R_n - \mathbf{A}_c)/2$			8d

4 CASCADE CALCULATIONS

4.1 CASCADING WITH CHAIN PARAMETERS

An advantage of using chain parameters for representing multi-ports lays in the fact that a cascade calculation is simply a matrix multiplication of the involved chain parameters. This follows directly from their definition and how the input and output signals are grouped in equation (7) and (8). The associated algorithm can be expressed with both the full as well as the decomposed matrix parameters, where the one with full matrix parameters is very simple to express.

But the advantage of using chain parameters for cascade calculations is obvious from an analysis point of view but has several drawbacks from a pure numerical point of view. Such a cascade approach is not only inefficient from a computational point of view (requires additional transformations between $\mathbf{S}$ and $\mathbf{T}$) but also sensitive to significant round-off errors when devices have high insertion losses. Both drawbacks originate from the fact that the originating data is often expressed in S (or in Z or Y parameters) and also requires a back transformation at the end of the cascade calculation. If the insertion loss of the multi-port is high (say 120 dB), then the parameter values in $\mathbf{S}_{oi}$ and $\mathbf{S}_{io}$ become very low and most values in $\mathbf{T}$ will become very high as well. But when the cascaded T parameters are back-transformed from $\mathbf{T}$ into $\mathbf{S}$, the reverse transmission matrix $\mathbf{S}_{io}$ is to be evaluated as $\mathbf{S}_{io} = \mathbf{T}_r - \mathbf{T}_b \times inv(\mathbf{T}_f) \times \mathbf{T}_a$ and requires subtractions from very high values to obtain very small values. These subtractions are the source of significant round-off errors.

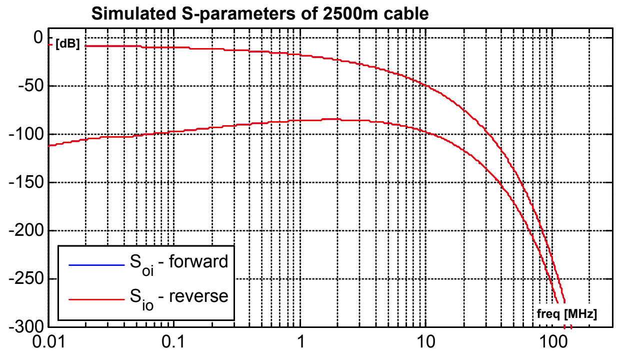

Fig. 1. Simulated forward (S_{oi}) and reverse (S_{io}) S-parameters of a double wire pair in a 2.5 km cable, representing transmission and FEXT.

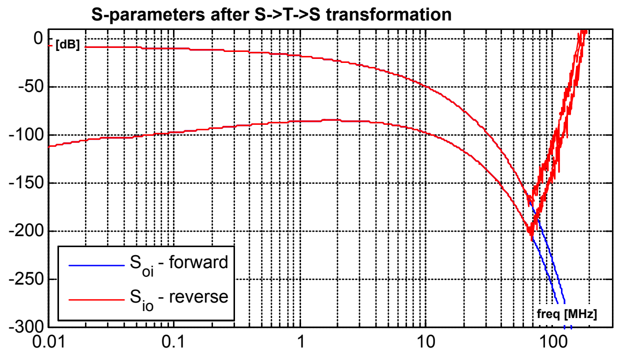

Fig. 2. Same curves as in figure 1, but now after an $\mathbf{S} \Rightarrow \mathbf{T} \Rightarrow \mathbf{S}$ transformation. Both figures should have shown the same results, but such a transformation causes significant round-off errors in reverse direction ($\mathbf{S}_{io}$) above 70 MHz.

Figure 1 and 2 illustrate how significant these round-off errors may be in practice. Both show the simulated transmission and far-end crosstalk (FEXT) up to 300MHz in forward as well as reverse direction through a double wire pair in a 2500m telephony cable. Forward and reverse curves are the same, as shown in figure 1, but figure 2 illustrate that they become significantly different above 70MHz after a parameter transformation $\mathbf{S} \Rightarrow \mathbf{T}$ to chain parameters, followed by a back transformation $\mathbf{T} \Rightarrow \mathbf{S}$ to scatter parameters. In general, when the insertion loss gets very high then the round-off errors cause a significant damage in reverse direction ($\mathbf{S}_{io}$).

These errors demonstrate the drawback of cascade calculations using T-parameters (or A-parameters) as intermediate step, especially because they can be avoided with another approach. Of course, we could have applied a-priory knowledge here on reciprocity of the cable, causing $\mathbf{S}_{io}$ to be equal to its transpose $\mathbf{S}_{oi}^{\mathsf{T}}$, but this would restrict the general use of cascade calculations.

4.2 CASCADING WITH S,Z,Y-PARAMETERS

The above mentioned numerical round-off errors by avoided all transformations between $\mathbf{S}$ and $\mathbf{T}$ as well. This means an alternative cascade algorithm that is expressed in only S-parameters and has no subtraction of huge values to obtain small ones. It is inconvenient to express such an algorithm directly in $\mathbf{S}$ but it is far more convenient to express that via its decomposed sub-matrices $\{\mathbf{S}_{ii}, \mathbf{S}_{oi}, \mathbf{S}_{io}, \mathbf{S}_{oo}\}$. Table 2 summarizes the cascade equations between decomposed matrix parameters. On input, the algorithm requires the sets of decomposed parameters of both multi-ports, such as $\{\mathbf{S1}_{ii}, \mathbf{S1}_{oi}, \mathbf{S1}_{io}, \mathbf{S1}_{oo}\}$ and $\{\mathbf{S2}_{ii}, \mathbf{S2}_{oi}, \mathbf{S2}_{io}, \mathbf{S2}_{oo}\}$. On output, it will produce a set of decomposed S-parameters, $\{\mathbf{S}_{ii}, \mathbf{S}_{oi}, \mathbf{S}_{io}, \mathbf{S}_{oo}\}$, that will represent the cascaded multi-port. And the same applies for $\mathbf{Z}$ or $\mathbf{Y}$ if voltages and currents are used.

TABLE 2
Cascade equations to evaluate the four decomposition matrices, for S,Z and Y-parameters.

$\mathbf{S}_{QF} = \mathbf{S2}_{oi} \times inv(\mathbb{I} - \mathbf{S1}_{oo} \times \mathbf{S2}_{ii})$	1x
$\mathbf{S}_{QR} = \mathbf{S1}_{io} \times inv(\mathbb{I} - \mathbf{S2}_{ii} \times \mathbf{S1}_{oo})$	1y
$\mathbf{S}_{ii} = \mathbf{S1}_{ii} + \mathbf{S}_{QR} \times \mathbf{S2}_{ii} \times \mathbf{S1}_{oi}$	1a
$\mathbf{S}_{oo} = \mathbf{S2}_{oo} + \mathbf{S}_{QF} \times \mathbf{S1}_{oo} \times \mathbf{S2}_{io}$	1b
$\mathbf{S}_{oi} = \mathbf{S}_{QF} \times \mathbf{S1}_{oi}$	1c
$\mathbf{S}_{io} = \mathbf{S}_{QR} \times \mathbf{S2}_{io}$	1d

$\mathbf{Y}_{Q} = inv(\mathbf{Z}1_{oo} + \mathbf{Z}2_{ii})$	2x
$\mathbf{Z}_{ii} = \mathbf{Z}1_{ii} - \mathbf{Z}1_{io} \times \mathbf{Y}_{Q} \times \mathbf{Z}1_{oi}$	2a
$\mathbf{Z}_{oo} = \mathbf{Z}2_{oo} - \mathbf{Z}2_{oi} \times \mathbf{Y}_{Q} \times \mathbf{Z}2_{io}$	2b
$\mathbf{Z}_{io} = \mathbf{Z}1_{io} \times \mathbf{Y}_{Q} \times \mathbf{Z}2_{io}$	2c
$\mathbf{Z}_{oi} = \mathbf{Z}2_{oi} \times \mathbf{Y}_{Q} \times \mathbf{Z}1_{oi}$	2d

$\mathbf{Z}_{Q} = inv(\mathbf{Y}2_{ii} + \mathbf{Y}1_{oo})$	3x
$\mathbf{Y}_{ii} = \mathbf{Y}1_{ii} - \mathbf{Y}1_{io} \times \mathbf{Z}_{Q} \times \mathbf{Y}1_{oi}$	1a
$\mathbf{Y}_{oo} = \mathbf{Y}2_{oo} - \mathbf{Y}2_{oi} \times \mathbf{Z}_{Q} \times \mathbf{Y}2_{io}$	1b
$\mathbf{Y}_{io} = -\mathbf{Y}1_{io} \times \mathbf{Z}_{Q} \times \mathbf{Y}2_{io}$	1c
$\mathbf{Y}_{oi} = -\mathbf{Y}2_{oi} \times \mathbf{Z}_{Q} \times \mathbf{Y}1_{oi}$	1d

The use of these alternative cascade expressions has proven to be quite robust. Even when we evaluate a huge chain of cable segments, for instance each representing 10cm of cable length, to calculate the s-parameters of a 1km telephony cable, then we will still get the same results as modeling 1km directly (as already depicted in figure 1). Cascade calculations according to the expressions in table 2 will then produce no round-off errors of any significance.

5 RECIPROCITY AND SYMMETRY

Another advantage of using decomposed matrices is that multi-port properties like reciprocity and/or symmetry translate into simple properties for the decomposed matrices. Reciprocity is a well-known property, that holds for all passive linear devices, and causes the full S-matrix to be equal to its transpose. Using this as a-priory knowledge, we can simplify algorithms where it applies. If we take that as starting point, we can derive all reciprocity relations summarized in table 3.

Symmetry means that the multi-port characteristics do not change when we swap a set of input ports with output ports. If symmetry applies to a particular multi-port then the equations in table 1 and 2 can be simplified by combining them with those in table 4. The matrix identities for T-parameters are equivalent with those summarized in section 3 of [14].

If both symmetry and reciprocilty applies then the equations can be simplified by using those in table 5, in addition to those in table 3 and 4.

TABLE 3
Matrix properties of multi-ports that are reciprocal.

reciprocity					
$\mathbf{S} = \mathbf{S}^\mathsf{T}$	1a	$\mathbf{Z} = \mathbf{Z}^\mathsf{T}$	2a	$\mathbf{Y} = \mathbf{Y}^\mathsf{T}$	3a
$\mathbf{S}_{oi} = \mathbf{S}_{io}^\mathsf{T}$	1b	$\mathbf{Z}_{oi} = \mathbf{Z}_{io}^\mathsf{T}$	2b	$\mathbf{Y}_{oi} = \mathbf{Y}_{io}^\mathsf{T}$	3b
$\mathbf{S}_{ii} = \mathbf{S}_{ii}^\mathsf{T}$	1c	$\mathbf{Z}_{ii} = \mathbf{Z}_{ii}^\mathsf{T}$	2c	$\mathbf{Y}_{ii} = \mathbf{Y}_{ii}^\mathsf{T}$	3b
$\mathbf{S}_{oo} = \mathbf{S}_{oo}^\mathsf{T}$	1d	$\mathbf{Z}_{oo} = \mathbf{Z}_{oo}^\mathsf{T}$	2d	$\mathbf{Y}_{oo} = \mathbf{Y}_{oo}^\mathsf{T}$	3d
$\mathbf{T}_a \times \mathbf{T}_f^\mathsf{T} - \mathbf{T}_f \times \mathbf{T}_a^\mathsf{T} = \mathbb{O}$					4a
$\mathbf{T}_b \times \mathbf{T}_r^\mathsf{T} - \mathbf{T}_r \times \mathbf{T}_b^\mathsf{T} = \mathbb{O}$					4b
$\mathbf{T}_f \times \mathbf{T}_r^\mathsf{T} - \mathbf{T}_a \times \mathbf{T}_b^\mathsf{T} = \mathbb{I}$					4c
$\mathbf{T}_r \times \mathbf{T}_f^\mathsf{T} - \mathbf{T}_b \times \mathbf{T}_a^\mathsf{T} = \mathbb{I}$					4e
$\begin{bmatrix} \mathbf{T_f} & \mathbf{T_a} \\ \mathbf{T_b} & \mathbf{T_r} \end{bmatrix} \times \begin{bmatrix} \mathbf{T_r}^\mathsf{T} & -\mathbf{T_a}^\mathsf{T} \\ -\mathbf{T_b}^\mathsf{T} & \mathbf{T_f}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{bmatrix}$					4f
$\mathbf{A}_z \times \mathbf{A}_v^\mathsf{T} - \mathbf{A}_v \times \mathbf{A}_z^\mathsf{T} = \mathbb{O}$					5a
$\mathbf{A}_y \times \mathbf{A}_c^\mathsf{T} - \mathbf{A}_c \times \mathbf{A}_y^\mathsf{T} = \mathbb{O}$					5b
$\mathbf{A}_v \times \mathbf{A}_c^\mathsf{T} - \mathbf{A}_z \times \mathbf{A}_y^\mathsf{T} = \mathbb{I}$					5c
$\mathbf{A}_c \times \mathbf{A}_v^\mathsf{T} - \mathbf{A}_y \times \mathbf{A}_z^\mathsf{T} = \mathbb{I}$					5d
$\begin{bmatrix} \mathbf{A_v} & \mathbf{A_z} \\ \mathbf{A_y} & \mathbf{A_c} \end{bmatrix} \times \begin{bmatrix} \mathbf{A_c}^\mathsf{T} & -\mathbf{A_z}^\mathsf{T} \\ -\mathbf{A_y}^\mathsf{T} & \mathbf{A_v}^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{bmatrix}$					5e

TABLE 4
Matrix properties of multi-ports that are symmetrical.

symmetry					
$\mathbf{S}_{oi} = \mathbf{S}_{io}$	1a	$\mathbf{Z}_{oi} = \mathbf{Z}_{io}$	2a	$\mathbf{Y}_{oi} = \mathbf{Y}_{io}$	3a
$\mathbf{S}_{ii} = \mathbf{S}_{oo}$	1b	$\mathbf{Z}_{ii} = \mathbf{Z}_{oo}$	2b	$\mathbf{Y}_{ii} = \mathbf{Y}_{oo}$	3b
$\mathbf{T}_a \times \mathbf{T}_f + \mathbf{T}_f \times \mathbf{T}_b = \mathbb{O}$					4a
$\mathbf{T}_b \times \mathbf{T}_r + \mathbf{T}_r \times \mathbf{T}_a = \mathbb{O}$					4b
$\mathbf{T}_f \times \mathbf{T}_r + \mathbf{T}_a \times \mathbf{T}_a = \mathbb{I}$					4c
$\mathbf{T}_r \times \mathbf{T}_f + \mathbf{T}_b \times \mathbf{T}_b = \mathbb{I}$					4d
$\begin{bmatrix} \mathbf{T_f} & \mathbf{T_a} \\ \mathbf{T_b} & \mathbf{T_r} \end{bmatrix} \times \begin{bmatrix} \mathbf{T_r} & \mathbf{T}_b \\ \mathbf{T_a} & \mathbf{T_f} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{bmatrix}$					4e
$\mathbf{A}_z \times \mathbf{A}_c - \mathbf{A}_v \times \mathbf{A}_z = \mathbb{O}$					5a
$\mathbf{A}_y \times \mathbf{A}_v - \mathbf{A}_c \times \mathbf{A}_y = \mathbb{O}$					5b
$\mathbf{A}_v \times \mathbf{A}_v - \mathbf{A}_z \times \mathbf{A}_y = \mathbb{I}$					5c
$\mathbf{A}_c \times \mathbf{A}_c - \mathbf{A}_y \times \mathbf{A}_z = \mathbb{I}$					5d
$\begin{bmatrix} \mathbf{A_v} & \mathbf{A_z} \\ \mathbf{A_y} & \mathbf{A_c} \end{bmatrix} \times \begin{bmatrix} \mathbf{A_v} & -\mathbf{A_z} \\ -\mathbf{A_y} & \mathbf{A_c} \end{bmatrix} = \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathbb{I} \end{bmatrix}$					5e

TABLE 5
Matrix properties of multi-ports that are reciprocal and symmetrical.

reciprocity & symmetry			
$\mathbf{T}_f = \mathbf{T}_f^\mathsf{T}$	1a	$\mathbf{A}_z = \mathbf{A}_z^\mathsf{T}$	2a
$\mathbf{T}_r = \mathbf{T}_r^\mathsf{T}$	1b	$\mathbf{A}_y = \mathbf{A}_y^\mathsf{T}$	2b
$\mathbf{T}_a = -\mathbf{T}_b^\mathsf{T}$	1c	$\mathbf{A}_v = +\mathbf{A}_c^\mathsf{T}$	2c

6 CONCLUSIONS

Although it is a common approach to use matrix methods for multi-ports analysis, it is not trivial how to bring them all together in a multi-port software toolbox in a generic, systematic and elegant way. Therefore we have defined a solid framework to do all kinds of matrix transformations and cascade calculations in a unified way. It is a matter of preference to perform these calculations with wave pairs or with voltage/current pairs, and therefore they are all treated equally all over this paper.

We have demonstrated that straight-forward cascade calculations via matrix multiplication of chain parameters can easily result in significant round-off errors. This may become a problem when calculating the cascade of cable sections into a cable with very high insertion loss. Therefore we proposed in table 2 a far more robust set of algorithms to calculate the cascade of multi-ports, and these algorithms are defined directly in decomposed S-, Z- or Y-parameters. The approach is robust enough for simulating a cascade of thousands of cable sections, without introducing round-off errors of any concern.

The derivation of the equations in this paper is quite elaborated, but the verification of them is simple. Just use complex random numbers for the s-parameters and verify that all transformations are consistent with each other.

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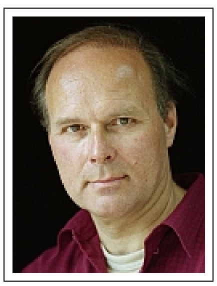

Rob F.M. van den Brink graduated in Electronics from Delft University in 1984, and received his PhD in 1994. He works as a senior scientist within TNO on broadband access networks. Since 1996, he has played a very prominent role in DSL standardization in Europe (ETSI, FSAN), has written more than 100 technical contributions to ETSI, and took the lead within ETSI-TM6 in identifying / defining cable models, test loops, noise models, performance tests, and spectral management. He is the editor of an ETSI-TM6 reference document on European cables, and led the creation of the MUSE Test Suite, a comprehensive document for analyzing access networks as a whole. He also designed solutions for Spectral Management policies in the Netherlands, and created various DSL tools for performance simulation (SPOCS, www.spocs.nl/en) and testing that are currently in the market. He has been Rapporteur/Editor for ETSI since 1999 (on Spectral Management: TR 101 830), Board Member of the MUSE consortium (2004-2008, www.ist-muse.org) and is now (since 2009) Work Package leader various projects of the Celtic 4GBB Consortium (2009-2017, www.4gbb.eu). This consortium initiated the development and standardization of G.fast.