Appendix

A. Transformation rules for two-port parameters

Two-port parameters (see section 2.2) are usually measured in s-parameters, normalized to a *real* reference impedance $Z_0=R_0=1/G_0$. The most relevant parameter transformations are therefore transformations from or to the s-parameter format. They are provided below.

For two-port matrix parameters (Z,Y,A,H,T,S) see section 2.3.1. For two-port virtual circuit parameters (BJT,FET) see section 2.3.2.

	from S	***to S (via Y)***
Z	$q_{sz} = ((s_{11}-1)\cdot(s_{22}-1)-s_{12}\cdot s_{21})/(2\cdot R_0)$ $z_{11} = (1-s_{22})/q_{sz} - R_0$ $z_{22} = (1-s_{11})/q_{sz} - R_0$ $z_{21} = s_{21}/q_{sz}$ $z_{12} = s_{12}/q_{sz}$	$q_{zs} = ((z_{11}+R_0)\cdot(z_{22}+R_0)-z_{12}\cdot z_{21})/(2\cdot R_0)$ $s_{11} = -(R_0+z_{22})/q_{zs} + 1$ $s_{22} = -(R_0+z_{11})/q_{zs} + 1$ $s_{21} = z_{21}/q_{zs}$ $s_{12} = z_{12}/q_{zs}$
Y	$q_{sy} = ((s_{11}+1)\cdot(s_{22}+1)-s_{21}\cdot s_{12})\cdot(R_0/2)$ $y_{11} = (1+s_{22})/q_{sy} - 1/R_0$ $y_{22} = (1+s_{11})/q_{sy} - 1/R_0$ $y_{21} = -s_{21}/q_{sy}$ $y_{12} = -s_{12}/q_{sy}$	$q_{ys} = ((y_{11}+G_0)\cdot(y_{22}+G_0)-y_{21}\cdot y_{12})\cdot(R_0/2)$ $s_{11} = (G_0+y_{22})/q_{ys} - 1$ $s_{22} = (G_0+y_{11})/q_{ys} - 1$ $s_{21} = -y_{21}/q_{ys}$ $s_{12} = -y_{12}/q_{ys}$
A	$q_{sa} = ((s_{11}+1)\cdot(s_{22}+1) - s_{12}\cdot s_{21})/2$ $a_{11} = (1 + s_{11}-q_{sa})/s_{21}$ $a_{22} = (1 + s_{22}-q_{sa})/s_{21}$ $a_{21} = (-s_{11}-s_{22}+q_{sa})/s_{21}/R_0$ $a_{12} = (q_{sa})/s_{21}\cdot R_0$	$q_{as} = (a_{11}+a_{12}/R_0+a_{21}\cdot R_0+a_{22})/2$ $s_{11} = (a_{11}+a_{12}/R_0)/q_{as} - 1$ $s_{22} = (a_{22}+a_{12}/R_0)/q_{as} - 1$ $s_{21} = 1/q_{as}$ $s_{12} = (a_{11}\cdot a_{22}-a_{21}\cdot a_{12})/q_{as}$
H	$q_{sh} = ((s_{11}-1)\cdot(s_{22}+1) - s_{12}\cdot s_{21})/2$ $h_{11} = -((1 + s_{22})/q_{sh} + 1)\cdot R_0$ $h_{22} = -((1 - s_{11})/q_{sh} + 1)/R_0$ $h_{21} = +s_{21}/q_{sh}$ $h_{12} = -s_{12}/q_{sh}$	$q_{hs} = ((s_{11}/R_0+1)\cdot(s_{22}\cdot R_0+1) - s_{12}\cdot s_{21})/2$ $s_{11} = -(1 + h_{22}\cdot R_0)/q_{hs} + 1$ $s_{22} = +(1 + h_{11}/R_0)/q_{hs} - 1$ $s_{21} = -h_{21}/q_{hs}$ $s_{12} = +h_{12}/q_{hs}$
T	$\Delta_s = (s_{11}\cdot s_{22} - s_{21}\cdot s_{12})$ $t_{11} = 1/s_{21}$ $t_{22} = -\Delta_s/s_{21}$ $t_{21} = +s_{11}/s_{21}$ $t_{12} = -s_{22}/s_{21}$	$\Delta_t = (t_{11}\cdot t_{22} - t_{21}\cdot t_{12})$ $s_{11} = t_{21}/t_{11}$ $s_{22} = -t_{12}/t_{11}$ $s_{21} = 1/t_{11}$ $s_{12} = \Delta_t/t_{11}$
B J T	$q_{sy} = ((s_{11}+1)\cdot(s_{22}+1)-s_{21}\cdot s_{12})\cdot(R_0/2)$ $\alpha_e = (s_{12}-s_{21})/(1+s_{22}-s_{12}-q_{sy}/R_0)$ $z_e = q_{sy}/(1+s_{22}-s_{21}-q_{sy}/R_0)$ $z_{ce} = q_{sy}/(1+s_{11}-s_{12}-q_{sy}/R_0)$ $z_{cb} = q_{sy}/(s_{12})$	$y_{11} = 1/(1+\alpha_e)/z_e + 1/z_{cb}$ $y_{22} = 1/z_{ce} + 1/z_{cb}$ $y_{21} = \alpha_e/(1+\alpha_e)/z_e - 1/z_{cb}$ $y_{12} = - 1/z_{cb}$
F E T	$q_{sy} = ((s_{11}+1)\cdot(s_{22}+1)-s_{21}\cdot s_{12})\cdot(R_0/2)$ $g_m = (s_{12}-s_{21})/q_{sy}$ $z_{gs} = q_{sy}/(1+s_{22}-s_{12}-q_{sy}/R_0)$ $z_{ds} = q_{sy}/(1+s_{11}-s_{12}-q_{sy}/R_0)$ $z_{dg} = q_{sy}/(s_{12})$	$y_{11} = 1/z_{gs} + 1/z_{dg}$ $y_{22} = 1/z_{ds} + 1/z_{dg}$ $y_{21} = g_m - 1/z_{dg}$ $y_{12} = - 1/z_{dg}$

$G_0=1/R_0=1/Z_0$

Appendix

B. Algorithms for overdetermined matrix divisions

Matrix division is a generalization of matrix inversion, and is used in many chapters of this manuscript. It is defined for square as well as rectangular matrices, provided that the associated matrix dimensions match. In all probability, the definition originates from the MATLAB©[123] manuals, although it should be defined in any standard textbook discussion on linear algebra. To simplify matters,

$\mathbf{A}\backslash\mathbf{B} \approx (\mathbf{A}^{-1})\cdot\mathbf{B}$ left-hand matrix division

$\mathbf{A}/\mathbf{B} \approx \mathbf{A}\cdot(\mathbf{B}^{-1})$ right-hand matrix division

The advantage of using division instead of inversion is that the concept is more general. It simplifies mathematical expressions when solving *overdetermined* sets of linear equations. This appendix describes the mathematical background of matrix division.

B.1. Definition of general matrix division.

The general definition is somewhat complicated, and is not representative for the way the division is evaluated (see section B.5) The division between two matrices is defined as:

$$\boxed{\begin{array}{l}\mathbf{A}\backslash\mathbf{b} \stackrel{\text{def}}{=} (\mathbf{A}'\cdot\mathbf{A})^{-1}\cdot(\mathbf{A}'\cdot\mathbf{b}) \\ \mathbf{c}/\mathbf{A} \stackrel{\text{def}}{=} (\mathbf{c}\cdot\mathbf{A}')\cdot(\mathbf{A}\cdot\mathbf{A}')^{-1}\end{array}} \Rightarrow \mathbf{A}\backslash\mathbf{b} = (\mathbf{b}'/\mathbf{A}')'$$

$\mathbf{A}' \equiv conj(\mathbf{A}^{\mathrm{T}})$ = conjugated transpose

The definition holds for square as well for rectangular matrices, provided that the associated matrix dimensions match. Its validation includes complex matrices. The next subsections will illustrate the meaning of these definitions.

B.2. Exact division of a square matrix

A classic application of matrix division and matrix inversion is finding a solution of a set of linear equations. These equation sets, have $\mathbf{A}\cdot\mathbf{z} = \mathbf{b}$ as general form. In this equation represent matrix $\mathbf{A}$ and $\mathbf{b}$ various equation constants and is $\mathbf{z}$ the solution vector. Matrix $\mathbf{A}$ must be square and invertable to enable $\mathbf{z} = \mathbf{A}^{-1}\cdot\mathbf{b}$.

In this special case the general definitions of matrix division simplifies significantly since $(\mathbf{A}'\cdot\mathbf{A})^{-1}\cdot\mathbf{A}' \equiv \mathbf{A}^{-1}$. This simplification emphasizes that matrix division is a generalization of matrix inversion because:

$\mathbf{A}\backslash\mathbf{b} = (\mathbf{A}^{-1})\cdot\mathbf{b}$

$\mathbf{c}/\mathbf{A} = \mathbf{c}\cdot(\mathbf{A}^{-1})$

$\mathbf{c}\cdot(\mathbf{A}\backslash\mathbf{b}) = \mathbf{c}\cdot(\mathbf{A}^{-1})\cdot\mathbf{b} = (\mathbf{c}/\mathbf{A})\cdot\mathbf{b}$

> The vector $\mathbf{z}_1=\mathbf{A}\backslash\mathbf{b}$ is the solution of matrix equation $\mathbf{A}\cdot\mathbf{z}_1=\mathbf{b}$.
> The vector $\mathbf{z}_2=\mathbf{b}/\mathbf{A}$ is the solution of matrix equation $\mathbf{z}_2\cdot\mathbf{A}=\mathbf{b}$.

Many textbook discussions on linear algebra suggest that the evaluation of $\mathbf{A}\backslash\mathbf{B}$ requires the inversion of matrix $\mathbf{A}$ followed by a matrix product with $\mathbf{B}$. This is as inefficient as calculating: $8/4 = 8\cdot(1/4) = 8\cdot(0.25) = 2$. Software packages, such as Matlab or LinPack

are provided with routines to calculate $\mathbf{A}\backslash\mathbf{B}$ more efficient than calculating $(\mathbf{A}^{-1})\cdot\mathbf{B}$ (see section B.5).

B.3. Least squares overdetermined division of a rectangular matrix

An important application of the generalized matrix division is finding a fair solution for overdetermined sets of equations. An overdetermined equation set, has $\mathbf{A}\cdot\mathbf{z} \approx \mathbf{b}$ as general form, and has no exact 'solution'. In this 'equation' represent matrix **A** and **b** various equation constants and **z** one of the many 'solution' vectors. Matrix **A** is a rectangular matrix.

The usefulness of 'solutions' **z** is closely related with the deviation $\Delta\mathbf{b}$ of vector $\mathbf{A}\cdot\mathbf{z}$ from vector **b**. This deviation can be analyzed as follows:

$$\mathbf{A}\cdot\mathbf{z} = \mathbf{b} + \Delta\mathbf{b}$$

let $\Delta\mathbf{z} \stackrel{\text{def}}{=} \mathbf{A}\backslash\mathbf{b} - \mathbf{z}$ = an offset vector of solution **z** relative to vector $\mathbf{A}\backslash\mathbf{b}$

let $\beta \stackrel{\text{def}}{=} \mathbf{A}\cdot(\mathbf{A}\backslash\mathbf{b})-\mathbf{b}$ = a substitution vector to simplify the equations

$$\Delta\mathbf{b} = \mathbf{A}\cdot\mathbf{z} - \mathbf{b}$$
$$\Delta\mathbf{b} = \mathbf{A}\cdot(\mathbf{A}\backslash\mathbf{b} - \Delta\mathbf{z}) - \mathbf{b}$$
$$\Delta\mathbf{b} = \mathbf{A}\cdot(\mathbf{A}\backslash\mathbf{b}) - \mathbf{A}\cdot(\Delta\mathbf{z}) - \mathbf{b}$$
$$\Delta\mathbf{b} = \beta - \mathbf{A}\cdot\Delta\mathbf{z}$$

The deviation vector $\Delta\mathbf{b}$ is a function of the offset vector $\Delta\mathbf{z}$. When using the relation $\mathbf{A}'\cdot\beta = \mathbf{0}$, which can be demonstrated simply from the definitions of β and $\mathbf{A}\backslash\mathbf{b}$, the value $|\Delta\mathbf{b}|^2$ equals:

$$|\Delta\mathbf{b}|^2 = (\beta - \mathbf{A}\cdot\Delta\mathbf{z})'\cdot(\beta - \mathbf{A}\cdot\Delta\mathbf{z})$$
$$|\Delta\mathbf{b}|^2 = \beta'\cdot\beta + (\mathbf{A}\cdot\Delta\mathbf{z})'\cdot(\mathbf{A}\cdot\Delta\mathbf{z}) - \beta'\cdot(\mathbf{A}\cdot\Delta\mathbf{z}) - (\mathbf{A}\cdot\Delta\mathbf{z})'\cdot\beta$$
$$|\Delta\mathbf{b}|^2 = |\beta|^2 + |\mathbf{A}\cdot\Delta\mathbf{z}|^2 - (\beta'\cdot\mathbf{A}\cdot\Delta\mathbf{z}) - (\Delta\mathbf{z}'\cdot\mathbf{A}'\cdot\beta)$$
$$|\Delta\mathbf{b}|^2 = |\beta|^2 + |\mathbf{A}\cdot\Delta\mathbf{z}|^2 - 0 - 0$$

This analysis shows that in all situations $|\Delta\mathbf{b}|^2 \geq |\beta|^2$, and that $|\Delta\mathbf{b}|^2$ is minimal when the offset vector $\Delta\mathbf{z}=\mathbf{0}$. This property makes $\mathbf{z}=\mathbf{A}\backslash\mathbf{b}$ often a fair choice, since it minimizes the 'error' in a least squares sense.

The solution $\mathbf{z}_1=\mathbf{A}\backslash\mathbf{b}$ of the matrix equation $\mathbf{A}\cdot\mathbf{z}_1=\mathbf{b}+\Delta\mathbf{b}_1$ minimizes $\lvert\Delta\mathbf{b}_1\rvert^2$
The solution $\mathbf{z}_2=\mathbf{b}/\mathbf{A}$ of the matrix equation $\mathbf{z}_2\cdot\mathbf{A}=\mathbf{b}+\Delta\mathbf{b}_2$ minimizes $\lvert\Delta\mathbf{b}_2\rvert^2$

These properties hold for real matrices as was for complex matrix equations (with complex solutions),

B.4. Least squares overdetermined division, restricted to real results

For many physical applications, complex solutions are excluded. Over-determined linear equation sets, have $\mathbf{A}\cdot\mathbf{x} \approx \mathbf{b}$ as general form, in which the matrices **A** and **b** may be *complex* provided that **x** is real. When complex solutions are excluded, the vector $\mathbf{A}\backslash\mathbf{b}$ is usually excluded to as solution.

The usefulness of real 'solutions' $\mathbf{x}$ is closely related with the complex deviation $\Delta\mathbf{b}$ of vector $\mathbf{A}\cdot\mathbf{x}$ from vector $\mathbf{b}$. This complex deviation can be transformed into a real format as follows:

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{b} + \Delta\mathbf{b}$$
$$(\mathbf{A}_{re} + j\cdot\mathbf{A}_{im})\cdot\mathbf{x} = (\mathbf{b}_{re} + j\cdot\mathbf{b}_{im}) + (\Delta\mathbf{b}_{re} + j\cdot\Delta\mathbf{b}_{im})$$
$$\begin{bmatrix}\mathbf{A}_{re}\\ \mathbf{A}_{im}\end{bmatrix}\cdot\mathbf{x} = \begin{bmatrix}\mathbf{b}_{re}\\ \mathbf{b}_{im}\end{bmatrix} + \begin{bmatrix}\Delta\mathbf{b}_{re}\\ \Delta\mathbf{b}_{im}\end{bmatrix}$$
$$\Rightarrow \left|\begin{matrix}\Delta\mathbf{b}_{re}\\ \Delta\mathbf{b}_{im}\end{matrix}\right|^2 = |\Delta\mathbf{b}_{re}|^2 + |\Delta\mathbf{b}_{im}|^2 = |\Delta\mathbf{b}_{re} + j\cdot\Delta\mathbf{b}_{im}|^2 = |\Delta\mathbf{b}|^2$$

This transformed form illustrates that vector $\Delta\mathbf{b}$ and the transformed variant of this vector have equal magnitude. This means that a real solution vector $\mathbf{x}$ that minimizes the transformed equation 'error' also minimizes the complex equation (within the restriction of real solutions).

The *real* solution vector $\mathbf{x}$ that minimizes $\|\Delta\mathbf{b}\|^2$ of the complex matrix equation $\mathbf{A}\cdot\mathbf{x}_1=\mathbf{b}+\Delta\mathbf{b}$, equals:	$\mathbf{x}_1 = \begin{bmatrix}\mathbf{A}_{re}\\ \mathbf{A}_{im}\end{bmatrix} \backslash \begin{bmatrix}\mathbf{b}_{re}\\ \mathbf{b}_{im}\end{bmatrix}$
The *real* solution vector $\mathbf{x}$ that minimizes $\|\Delta\mathbf{c}\|^2$ of the complex matrix equation $\mathbf{x}_2\cdot\mathbf{A}=\mathbf{c}+\Delta\mathbf{c}$, equals:	$\mathbf{x}_2 = \begin{bmatrix}\mathbf{c}_{re}\\ \mathbf{c}_{im}\end{bmatrix} / \begin{bmatrix}\mathbf{A}_{re}\\ \mathbf{A}_{im}\end{bmatrix}$

B.5. Division algorithm

Although the definition of the matrix division has facilitated a direct evaluation, a more appropriate approach is recommended. In the general case that $\mathbf{A}$ is rectangular, the definition is reduced by means of the QR-decomposition [413,414] of matrix $\mathbf{A}$ in an orthogonal matrix $\mathbf{Q}$ and an upper triangle matrix $\mathbf{R}$.
Because $\mathbf{A} = \mathbf{Q}\cdot\mathbf{R}$ and $\mathbf{Q}'\cdot\mathbf{Q} = \mathbf{1}$ by definition, the evaluation of the matrix division reduces to:

$$\begin{aligned}\mathbf{A}\backslash\mathbf{b} &= (\mathbf{R}'\cdot\mathbf{Q}' \cdot \mathbf{Q}\cdot\mathbf{R})^{-1}\cdot(\mathbf{R}'\cdot\mathbf{Q}'\cdot\mathbf{b})\\ &= (\mathbf{R}'\cdot\mathbf{R})^{-1}\cdot(\mathbf{R}'\cdot\mathbf{Q}')\cdot\mathbf{b}\\ &= \mathbf{R}\backslash(\mathbf{Q}'\cdot\mathbf{b})\end{aligned}$$

The benefit of this approach is that the upper triangle form of $\mathbf{R}$ dramatically simplifies the division, because many elements are zero. If back substitution ([413:section 2.2] is performed, then an explicit division with $\mathbf{R}$ is not required. Further, a robust QR-factorization, based on Householder transformations ([413:section 11.2, 414] reduces numerical round-off errors for ill-conditioned matrices $\mathbf{A}$. And finally, the explicit evaluation of $\mathbf{Q}$ is not required if each Householder transformation step, that reduces $\mathbf{A}$ toward $\mathbf{R}$, is simultaneously performed on vector $\mathbf{b}$.

Appendix

C. Algorithm for polynomial curve fits

This appendix discusses an algorithm for finding the *real* coefficients of a polynomial that optimally fits to a table with measured data. The polynomial represents a transfer function of the following general form:

$$H(j\omega) = H(s) \stackrel{def}{=} a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_n \cdot s^n$$

When an *exact* fit exists it is irrelevant how the coefficients $[a_0,a_1,a_2,...a_n]$ are calculated, since that solution is unique. When H(s) is measured at k frequencies ($k>n$) numerous 'solutions' exist. We call a solution *optimal* when the *relative* error δ of H(s) is minimized, rms-averaged over all frequencies of interest.
Let h denote the measured function value of H(s) at complex frequency $s=j\omega$, then:

$$a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_n \cdot s^n \approx h$$

$$a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_n \cdot s^n = h \cdot (1+\delta)$$

$$a_0/h + a_1 \cdot s/h + a_2 \cdot s^2/h + ... + a_n \cdot s^n/h = (1+\delta)$$

$$a_0 \cdot (w/h) + a_1 \cdot s \cdot (w/h) + a_2 \cdot s^2 \cdot (w/h) + ... + a_n \cdot s^n \cdot (w/h) = w \cdot (1+\delta)$$

The quantity w represents an arbitrary weighing factor, to be discussed below. To simplify these equations, the vectors $\boldsymbol{\alpha}$, $\mathbf{R}$, $\Delta\mathbf{R}$, and $\mathbf{Q}$ are as follows defined for all frequencies in the table:

$$\boldsymbol{\alpha} \stackrel{def}{=} \begin{bmatrix} a_0 \\ a_1 \\ ... \\ a_n \end{bmatrix} \qquad \mathbf{R} \stackrel{def}{=} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ ... \\ w_k \end{bmatrix} \qquad \Delta\mathbf{R} \stackrel{def}{=} \begin{bmatrix} w_1 \cdot \delta_1 \\ w_2 \cdot \delta_2 \\ w_3 \cdot \delta_3 \\ ... \\ w_k \cdot \delta_k \end{bmatrix}$$

$$\mathbf{Q} \stackrel{def}{=} \begin{bmatrix} w_1/h_1 & w_1/h_1 \cdot s_1 & w_1/h_1 \cdot s_1^2 & ... & w_1/h_1 \cdot s_1^n \\ w_2/h_2 & w_2/h_2 \cdot s_2 & w_2/h_2 \cdot s_2^2 & ... & w_2/h_2 \cdot s_2^n \\ w_3/h_3 & w_3/h_3 \cdot s_3 & w_3/h_3 \cdot s_3^2 & ... & w_3/h_3 \cdot s_3^n \\ ... & ... & ... & ... & ... \\ w_k/h_k & w_k/h_k \cdot s_k & w_k/h_k \cdot s_k^2 & ... & w_k/h_k \cdot s_k^n \end{bmatrix}$$

$$\mathbf{Q} \cdot \boldsymbol{\alpha} = \mathbf{R} + \Delta\mathbf{R}$$

$$\boldsymbol{\alpha} = \begin{bmatrix} \mathbf{Q}_{re} \\ \mathbf{Q}_{im} \end{bmatrix} \backslash \begin{bmatrix} \mathbf{R}_{re} \\ \mathbf{R}_{im} \end{bmatrix}$$

see appendix ***B*** for the definition of an over-determined left-hand matrix division, restricted to real results

The vector $\boldsymbol{\alpha}$ is a solution that minimizes $\Delta\mathbf{R}$ in a least squares sense (see appendix *B*). This means that the rms-average of all weighed 'errors' $(w \cdot \delta)$ is minimized.
When the weighing factors w are *balanced*, meaning that they are all equal for all frequencies (e.g. $w \equiv 1$), the relative error δ is minimized and $\boldsymbol{\alpha}$ is the requested optimal solution. The use of *unbalanced* weighing factors w makes the fit more (or less) sensitive in restricted frequency ranges. In case $w=0$ in a frequency interval, the associated errors are completely ignored. This is handy to exclude spikes and other errors in measurements, or to exclude frequency ranges in which the transfer function is inadequately predicted by the chosen polynomial.

Appendix

D. Algorithm for rational curve fits

This appendix discusses an algorithm for finding the *real* coefficients of a rational function that optimally fits to a table with measured data. The rational function represents a transfer function of the following general form:

$$H(j\omega) = H(s) \stackrel{\text{def}}{=} g \cdot \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m}{b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n} = g \cdot \frac{T(s)}{N(s)}$$

When an *exact* fit exists it is irrelevant how the coefficients are calculated, since that solution is unique. When H(s) is measured at k frequencies ($k>n$) numerous 'solutions' exist. We call a solution *optimal* when the *relative* error δ of H(s) is minimized, rms-averaged over all frequencies of interest.
Let h denote the measured function value of H(s) at complex frequency $s=j\omega$, and w an arbitrary weighing factor as discussed in appendix *C* then:

$$g \cdot \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m}{b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n} \approx h$$

$$g \cdot \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m}{b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n} = h \cdot (1+\delta)$$

$$w \cdot g \cdot \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m}{b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n} = h \cdot w \cdot (1+\delta)$$

$$w \cdot g \cdot (a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m) = h \cdot w \cdot (1+\delta)(b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n)$$

$$\eta \stackrel{\text{def}}{=} N(s) \stackrel{\text{def}}{=} (b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n)$$

$$w \cdot g \cdot (a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m) - h \cdot w \cdot (b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1}) = h \cdot w \cdot (s^n + \delta \cdot \eta)$$

$$w/\eta \cdot g/h \cdot (a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m) - w/\eta \cdot (b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1}) = w/\eta \cdot s^n + w \cdot \delta$$

To simplify these equations, the vectors α, $\mathbf{R}$, $\Delta\mathbf{R}$ and $\mathbf{Q}$ are as follows defined for all frequencies in the table:

$$\alpha \stackrel{\text{def}}{=} \begin{bmatrix} g \cdot a_0 \\ g \cdot a_1 \\ ... \\ g \cdot a_{m-1} \\ g \\ b_0 \\ b_1 \\ ... \\ b_{n-1} \end{bmatrix} \qquad \mathbf{R} \stackrel{\text{def}}{=} \begin{bmatrix} w_1/\eta_1 \cdot s_1^n \\ w_2/\eta_2 \cdot s_2^n \\ w_3/\eta_3 \cdot s_3^n \\ ... \\ w_k/\eta_k \cdot s_k^n \end{bmatrix} \qquad \Delta\mathbf{R} \stackrel{\text{def}}{=} \begin{bmatrix} w_1 \cdot \delta_1 \\ w_2 \cdot \delta_2 \\ w_3 \cdot \delta_3 \\ ... \\ w_k \cdot \delta_k \end{bmatrix} \qquad \mathbf{W} \stackrel{\text{def}}{=} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ ... \\ w_k \end{bmatrix}$$

$$\mathbf{Q} \stackrel{\text{def}}{=} \begin{bmatrix} w_1/\eta_1/h_1 & w_1/\eta_1/h_1 \cdot s_1 & .... & w_1/\eta_1/h_1 \cdot s_1^m & -w_1/\eta_1 & -w_1/\eta_1 \cdot s_1 & .... & -w_1/\eta_1 \cdot s_1^{n-1} \\ w_2/\eta_2/h_2 & w_2/\eta_2/h_2 \cdot s_2 & .... & w_2/\eta_2/h_2 \cdot s_2^m & -w_2/\eta_2 & -w_2/\eta_2 \cdot s_2 & .... & -w_2/\eta_2 \cdot s_2^{n-1} \\ w_3/\eta_3/h_3 & w_3/\eta_3/h_3 \cdot s_3 & .... & w_3/\eta_3/h_3 \cdot s_3^m & -w_3/\eta_3 & -w_3/\eta_3 \cdot s_3 & .... & -w_3/\eta_3 \cdot s_3^{n-1} \\ .... & .... & .... & .... & .... & .... & .... & .... \\ w_k/\eta_k/h_k & w_k/\eta_k/h_k \cdot s_k & .... & w_k/\eta_k/h_k \cdot s_k^m & -w_k/\eta_k & -w_k/\eta_k \cdot s_k & .... & -w_k/\eta_k \cdot s_k^{n-1} \end{bmatrix}$$

$$\mathbf{Q} \cdot \alpha = \mathbf{R} + \Delta\mathbf{R}$$

$$\alpha = \begin{bmatrix} \mathbf{Q}_{re} \\ \mathbf{Q}_{im} \end{bmatrix} \backslash \begin{bmatrix} \mathbf{R}_{re} \\ \mathbf{R}_{im} \end{bmatrix}$$

see appendix *B* for the definition of an over-determined left-hand matrix division, restricted to real results

The vector α is a solution that minimizes $\Delta\mathbf{R}$ in a least squares sense (see appendix *B*). This means that the rms-average of all weighed 'errors' ($w \cdot \delta$) is minimized.
This approach of calculating α seems to be somewhat tricky because the calculation of the values η requires the coefficients of α as input. The essential point of this approach is that the numerical values of η are *estimated* for each frequency of the table. If an exact fit exists the solution α is independent of the estimated values η. If it does not exist, the estimated values of η are iterative improved until a stable solution vector α is found. The iteration algorithm is as symbolically illustrated below:

α = TransferFit(**h**, **s**, n,m, **w**)
h, **s** are column vectors with length *k* representing the (measured) transfer function values and the associated complex frequencies. The integers *n* and *m* represent the requested polynomial orders (number of poles and zeros). **w** is a column vector representing the weighing factors; when omitted choose **w**≡1
$s_{avg} = j \cdot \sqrt[k]{\lvert s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot \ldots \cdot s_k \rvert}$ {is the geometric mean of all complex frequencies $s=j\omega$} $\eta = \frac{1+(s/s_{avg})^m}{h}$ {is a fair initial guess} $\alpha_0 := \mathbf{0}$; *repeat* *evaluate* [**R**, **Q**] *from* [**h**,**s**,η,**w**] *evaluate* α *from* [**R**, **Q**] *evaluate* η *from* [α, **s** ,n, m] $\Delta\alpha := \alpha - \alpha_0$; $\alpha_0 := \alpha$ *until* $\lvert\Delta\alpha\rvert \ll \lvert\alpha\rvert$

An attractive point of this iteration is that it requires no starting values for α. On the other hand, it requires an initial guess for η, which is much simpler. This property makes it a robust and fast converging algorithm.

When the weighing factors w are *balanced*, meaning that they are all equal for all frequencies (e.g. w≡1), the relative error δ is minimized and α is the requested optimal solution. The use of *unbalanced* weighing factors w makes the iterative fit more (or less) sensitive in restricted frequency ranges. In case w=0 in a frequency interval, the associated errors are completely ignored. This is handy to exclude spikes and other errors in measurements, or to exclude frequency ranges in which the transfer function is inadequately predicted by the chosen rational function.

Appendix

E. Algorithm for rational magnitude fit

This appendix discusses an algorithm for finding the *real* coefficients of a rational function that optimally fits to a table with measured data, in a special case. The rational function represents a transfer function of the following general form:

$$H(j\omega) = H(s) \stackrel{\text{def}}{=} g \cdot \frac{a_0 + a_1 \cdot s + a_2 \cdot s^2 + ... + a_{m-1} \cdot s^{m-1} + s^m}{b_0 + b_1 \cdot s + b_2 \cdot s^2 + ... + b_{n-1} \cdot s^{n-1} + s^n} = g \cdot \frac{T(s)}{N(s)}$$

The general curve fit is discussed in appendix *D*. This appendix is focused on situations that the information on the transfer function is restricted to $|H(j\omega)|$ or $H(j\omega) / |H(j\omega)|$.

If the information of a rational transfer function $H(j\omega)$ with complex function values is restricted to the real values $|H(j\omega)|$, then the square of these real numbers is a rational function too. Using this property, the coefficients of this related rational function are evaluated as follows:

$$|H(j\omega)|^2 = H(+j\omega) \cdot H(-j\omega) = g^2 \cdot \frac{T(+j\omega)}{N(+j\omega)} \cdot \frac{T(-j\omega)}{N(-j\omega)}$$

$$|H(j\omega)|^2 \stackrel{\text{def}}{=} g^2 \cdot \frac{A_0 + A_2 \cdot s^2 + A_4 \cdot s^4 + ... + A_{2m-2} \cdot s^{2m-2} + s^{2m}}{B_0 + B_2 \cdot s^2 + B_4 \cdot s^4 + ... + B_{2n-2} \cdot s^{2n-2} + s^{2n}} \stackrel{\text{def}}{=} G \cdot \frac{T(s^2)}{N(s^2)}$$

$$\boldsymbol{\beta} \stackrel{\text{def}}{=} \begin{bmatrix} G \cdot A_0 \\ G \cdot A_2 \\ ... \\ G \cdot A_{2m-2} \\ G \\ B_0 \\ B_2 \\ ... \\ B_{2n-2} \end{bmatrix}$$

$\boldsymbol{\beta}$ = TransferFit($|\mathbf{h}|^2$, $\mathbf{s}^2$, n, m, **w**) see appendix *D* for the definition of function TransferFit

The coefficients in $\boldsymbol{\beta}$ generate a rational function that describes the function $H(s) \cdot H(-s)$. The poles and zeros of $H(+s)$ are symmetrical situated in the complex plane, with respect to the poles and zeros of $H(-s)$. There are no means available to distinct them in a pole-zero pattern of $H(s) \cdot H(-s)$. Nevertheless, a good choice is based on the assumption that $H(s)$ is a *minimum phase* transfer function.
$H(s)$ is reconstructed from the pole-zero pattern of $H(s) \cdot H(-s)$ by combining all left half-plane singularities and ejecting all other poles and zeros.

Appendix

F. Algorithm for rational delay fit

This appendix discusses an algorithm for finding the *real* coefficients of a rational function that optimally fits to a table with measured data, in a special case. The rational function represents a transfer function of the following general form:

$$H(j\omega) = H(s) \stackrel{\text{def}}{=} \frac{1 - a_1\cdot s + a_2\cdot s^2 - .... - a_{n-1}\cdot s^{n-1} + a_n\cdot s^n}{1 + a_1\cdot s + a_2\cdot s^2 + ... + a_{n-1}\cdot s^{n-1} + a_n\cdot s^n} = \frac{N(-s)}{N(+s)}$$

when the order n is odd, the last term in the denominator is negative

These rational functions have the property that $|H(j\omega)|\equiv 1$, which makes them attractive to model (pseudo) delay using a *finite* number of poles and zeros. The general curve fit is discussed in appendix *D*. This appendix is a variant of that general fit, and is focused on the differences only.

Let h denote the measured function value of H(s) at complex frequency $s=j\omega$, let w be an arbitrary weighing factor (see appendix *C*), and δ the relative error of h. Then:

$$\frac{1 - a_1\cdot s + a_2\cdot s^2 - .... - a_{n-1}\cdot s^{n-1} + a_n\cdot s^n}{1 + a_1\cdot s + a_2\cdot s^2 + ... + a_{n-1}\cdot s^{n-1} + a_n\cdot s^n} \approx h$$

$$w\cdot\frac{1 - a_1\cdot s + a_2\cdot s^2 - .... - a_{n-1}\cdot s^{n-1} + a_n\cdot s^n}{1 + a_1\cdot s + a_2\cdot s^2 + ... + a_{n-1}\cdot s^{n-1} + a_n\cdot s^n} = h\cdot w\cdot(1+\delta)$$

$$\eta \stackrel{\text{def}}{=} N(s) \stackrel{\text{def}}{=} (1+a_1\cdot s+ a_2\cdot s^2+ ..... +a_{n-1}\cdot s^{n-1} + a_{n-1}\cdot s^n)$$

$$w/h\cdot(1-a_1\cdot s+a_2\cdot s^2- .... -a_{n-1}\cdot s^{n-1}+a_n\cdot s^n)-w(1+a_1\cdot s+ a_2\cdot s^2+ ..... +a_{n-1}\cdot s^{n-1}+a_n\cdot s^n)= w\cdot\eta\cdot\delta$$

$$-w\cdot\{(1-1/h) + (1+1/h)\cdot a_1\cdot s + (1-1/h)\cdot a_2\cdot s^2 + ... + (1+1/h)\cdot a_{n-1}\cdot s^{n-1}\}= w\cdot\eta\cdot\delta$$

$$(w/\eta)\cdot\{(1+1/h)\cdot a_1\cdot s + (1-1/h)\cdot a_2\cdot s^2 + ... + (1+1/h)\cdot a_{n-1}\cdot s^{n-1}\}= (w/\eta)\cdot(1/h-1) - w\cdot\delta$$

To simplify these equations, the vectors α, $\mathbf{R}$, $\Delta\mathbf{R}$ and $\mathbf{Q}$ are as follows defined for all frequencies in the table:

$$\alpha \stackrel{\text{def}}{=} \begin{bmatrix} a_1 \\ a_2 \\ ... \\ a_n \end{bmatrix} \qquad \mathbf{R} \stackrel{\text{def}}{=} \begin{bmatrix} (1/h_1-1)\cdot w_1/\eta_1 \\ (1/h_2-1)\cdot w_2/\eta_2 \\ (1/h_3-1)\cdot w_3/\eta_3 \\ ... \\ (1/h_k-1)\cdot w_k/\eta_k \end{bmatrix} \qquad \Delta\mathbf{R} \stackrel{\text{def}}{=} \begin{bmatrix} w_1\cdot\delta_1 \\ w_2\cdot\delta_2 \\ w_3\cdot\delta_3 \\ ... \\ w_k\cdot\delta_k \end{bmatrix} \qquad \mathbf{W} \stackrel{\text{def}}{=} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ ... \\ w_k \end{bmatrix}$$

$$\mathbf{Q} \stackrel{\text{def}}{=} \begin{bmatrix} (1+1/h_1)\cdot w_1/\eta_1 & (1-1/h_1)\cdot w_1/\eta_1\cdot s_1 & (1+1/h_1)\cdot w_1/\eta_1\cdot s_1^2 & .... & (1+1/h_1)\cdot w_1/\eta_1\cdot s_1^{n-1} \\ (1+1/h_2)\cdot w_2/\eta_2 & (1-1/h_2)\cdot w_2/\eta_2\cdot s_2 & (1+1/h_2)\cdot w_2/\eta_2\cdot s_2^2 & .... & (1+1/h_2)\cdot w_2/\eta_2\cdot s_2^{n-1} \\ (1+1/h_3)\cdot w_3/\eta_3 & (1-1/h_3)\cdot w_3/\eta_3\cdot s_3 & (1+1/h_3)\cdot w_3/\eta_3\cdot s_3^2 & .... & (1+1/h_3)\cdot w_3/\eta_3\cdot s_3^{n-1} \\ .... & .... & .... & .... & .... \\ (1+1/h_k)\cdot w_k/\eta_k & (1-1/h_k)\cdot w_k/\eta_k\cdot s_k & (1+1/h_k)\cdot w_k/\eta_k\cdot s_k^2 & .... & (1+1/h_k)\cdot w_k/\eta_k\cdot s_k^{n-1} \end{bmatrix}$$

$$\mathbf{Q}\cdot\alpha = \mathbf{R} - \Delta\mathbf{R}$$

$$\alpha = \begin{bmatrix} \mathbf{Q}_{re} \\ \mathbf{Q}_{im} \end{bmatrix} \backslash \begin{bmatrix} \mathbf{R}_{re} \\ \mathbf{R}_{im} \end{bmatrix}$$

see appendix *B* for the definition of an over-determined left-hand matrix division, restricted to real results

Similarly to the algorithm in appendix *D*, the values of η must be estimated and improved in a similarly iterative way.

Appendix

G. Algorithm for weighed polynomial division

This appendix discusses a generalized algorithm for finding a quotient polynomial of two polynomials. If the rest of a polynomial division is zero, then the quotient polynomial is unique. For non-zero rest polynomials, different quotients exist.
Division of T(s)/N(s) results in some q(s) and r(s), that are as follows related to T(s) and N(s):

$T(s) = q_r(s) \cdot N(s) + r_r(s)$ *normal (backward) division with remainder*
$T(s) = q_f(s) \cdot N(s) + r_f(s) \cdot s^{m-k}$ *reverse (forward) division with remainder*
$T(s) = q_g(s) \cdot N(s) + r_g(s) \cdot W(s)$ *weighed division with remainder*

T(s) = numerator q(s) = quotient
N(s) = denominator r(s) = remainder
W(s) = weighing function

The weighing polynomial W(s) specifies what quotient is calculated. This appendix discusses an algorithm to evaluate {q(s), r(s)} when {T(s), N(s), W(s)} are specified.

When the polynomial coefficients of q(s), r(s) and T(s) are organized as column vectors, and N(s) and W(s) as matrices, then the weighed polynomial quotient can be calculated using matrix division (see appendix *B*). This is illustrated in the following example:

$$\begin{bmatrix} n0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n1 & n0 & 0 & 0 & 0 & 0 & 0 & 0 \\ n2 & n1 & n0 & 0 & 0 & 0 & 0 & 0 \\ n3 & n2 & n1 & 0 & 0 & 0 & 0 & 0 \\ n4 & n3 & n2 & 0 & 0 & 0 & 0 & 0 \\ n5 & n4 & n3 & 0 & 0 & 0 & 0 & 0 \\ 0 & n5 & n4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} q0 \\ q1 \\ q2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & w0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & w1 & w0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & w1 & w0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w1 & w0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & w1 & w0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ r0 \\ r1 \\ r2 \\ r3 \\ r4 \end{bmatrix} = \begin{bmatrix} t0 \\ t1 \\ t2 \\ t3 \\ t4 \\ t5 \\ t6 \\ t7 \end{bmatrix}$$

When the individual matrix products are combined, then:

$$\begin{bmatrix} n0 & 0 & 0 & w0 & 0 & 0 & 0 & 0 \\ n1 & n0 & 0 & w1 & w0 & 0 & 0 & 0 \\ n2 & n1 & n0 & 0 & w1 & w0 & 0 & 0 \\ n3 & n2 & n1 & 0 & 0 & w1 & w0 & 0 \\ n4 & n3 & n2 & 0 & 0 & 0 & w1 & w0 \\ n5 & n4 & n3 & 0 & 0 & 0 & 0 & w1 \\ 0 & n5 & n4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} q0 \\ q1 \\ q2 \\ r0 \\ r1 \\ r2 \\ r3 \\ r4 \end{bmatrix} = \begin{bmatrix} t0 \\ t1 \\ t2 \\ t3 \\ t4 \\ t5 \\ t6 \\ t7 \end{bmatrix}$$

$$\begin{bmatrix} q0 \\ q1 \\ q2 \\ r0 \\ r1 \\ r2 \\ r3 \\ r4 \end{bmatrix} = \begin{bmatrix} n0 & 0 & 0 & w0 & 0 & 0 & 0 & 0 \\ n1 & n0 & 0 & w1 & w0 & 0 & 0 & 0 \\ n2 & n1 & n0 & 0 & w1 & w0 & 0 & 0 \\ n3 & n2 & n1 & 0 & 0 & w1 & w0 & 0 \\ n4 & n3 & n2 & 0 & 0 & 0 & w1 & w0 \\ n5 & n4 & n3 & 0 & 0 & 0 & 0 & w1 \\ 0 & n5 & n4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & n5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \backslash \begin{bmatrix} t0 \\ t1 \\ t2 \\ t3 \\ t4 \\ t5 \\ t6 \\ t7 \end{bmatrix}$$

A general division algorithm must copy the polynomial coefficients of N(s),T(s),W(s) to matrices, in accordance with this example, followed by matrix division. The coefficients of the result matrix must be copied to the polynomials q(s) and r(s).

Appendix

H. Algorithm for dominant deflation of transfer order

This appendix discusses an algorithm to approximate a low-pass rational transfer function H(s) with another rational function $H_d(s)$ with lower transfer order. On input, the coefficients of H(s) are provided (or the poles and zeros) and the maximum frequency of interest: ω_0. On output, the coefficients of $H_d(s)$ are provided by the algorithm. By definition, $H(j\omega_0)=H_d(j\omega_0)$ and $H(j\omega)\approx H_d(j\omega)$ for $\omega<\omega_0$. An application example of this algorithm is loopgain deflation, as discussed in section 4.5

Rational functions are the quotient of two polynomials. Higher order polynomial terms are relevant for 'high' frequencies only, and can be omitted when irrelevant. The consequence of this type of deflation method is that the error of the deflated transfer function increases with the frequency. For loopgain deflation it is crucial that the deflated function value for $\omega=\omega_0$ equals the original function value. Therefore our deflation algorithm is performed in two steps:

- The first step removes all power terms that contribute less than a predefined value, e.g. 30%, for frequencies lower than the bandwidth ($\omega \le \omega_0$).
- The second step changes the two highest coefficients of numerator as well as denominator to match $H(j\omega_0)=H_d(j\omega_0)$.

We will illustrate this approach using an example of a transfer function with four zeros and five poles. H(s) is deflated into $H_d(s)$ using $H_0(s)$ as intermediate result. To simplify the description, the coefficients of the numerator and the denominator are scaled with respect to ω_0 (the corner frequency of the bandwidth of interest).

$$H(s) = G_0 \cdot \frac{1 + a_1\cdot(s/\omega_0) + a_2\cdot(s/\omega_0)^2 + a_3\cdot(s/\omega_0)^3 + a_4\cdot(s/\omega_0)^4}{1 + b_1\cdot(s/\omega_0) + b_2\cdot(s/\omega_0)^2 + b_3\cdot(s/\omega_0)^3 + b_4\cdot(s/\omega_0)^4 + b_5\cdot(s/\omega_0)^5}$$

$$H_0(s) = G_0 \cdot \frac{1 + a_1\cdot(s/\omega_0) + a_2\cdot(s/\omega_0)^2}{1 + b_1\cdot(s/\omega_0) + b_2\cdot(s/\omega_0)^2 + b_3\cdot(s/\omega_0)^3 + b_4\cdot(s/\omega_0)^4} \qquad \textit{step 1}$$

$$H_d(s) = G_0 \cdot \frac{1 + \alpha_1\cdot(s/\omega_0) + \alpha_2\cdot(s/\omega_0)^2}{1 + b_1\cdot(s/\omega_0) + b_2\cdot(s/\omega_0)^2 + \beta_3\cdot(s/\omega_0)^3 + \beta_4\cdot(s/\omega_0)^4} \qquad \textit{step 2}$$

step 1

The removal of power terms in step 1 is an iterative process. The algorithm removes the highest term when it fulfills the requirement that its magnitude at $\omega=\omega_0$ is a factor δ below the magnitude of the summated proceeding terms.

repeat
  if $|a_k| < \delta \cdot abs(1 + a_1 \cdot j + a_2 \cdot j^2 + a_3 \cdot j^3 + \ldots + a_{k-1} \cdot j^{k-1})$ *then* remove a_k
  if $|b_k| < \delta \cdot abs(1 + b_1 \cdot j + b_2 \cdot j^2 + b_3 \cdot j^3 + \ldots + b_{k-1} \cdot j^{k-1})$ *then* remove b_k
until nothing to remove

The iteration proceeds until the above criterion fails.

step 2

The first deflated approximation is exact for ω=0 however the accuracy decreases with increasing frequency. Step two of the deflation is improvement of the deflation accuracy for frequencies near $\omega=\omega_0$. The complex error in the polynomials of H(s) after step 1, and the proposed correction for step 2 is:

error in numerator when deflated from order M→m after removal of the associated higher order terms	modification of the highest coefficients in the deflated numerator
$\delta_a = \sum_{k=m+1}^{k=M}\left(a_k \cdot j^{(k-m)}\right)$	$\alpha_{m-1} = a_{m-1} - imag\{\delta_a\}$ $\alpha_m = a_m + real\{\delta_a\}$
error in denominator when deflated from order N→ n after removal of the associated higher order terms	modification of the highest coefficients in the deflated denominator
$\delta_b = \sum_{k=n+1}^{k=N}\left(b_k \cdot j^{(k-n)}\right)$	$\beta_{n-1} = a_{n-1} - imag\{\delta_b\}$ $\beta_n = a_n + real\{\delta_b\}$

In the special case that one of the deflated polynomials is a first order polynomial (or a constant), the proposed algorithm must perform a slightly different approach.
It requires the replacement of the correction pair $\{\delta_a, \delta_b\}$ by the factors δ_{a0} or δ_{b0} alone, as defined below:

$$H(j\omega_0) = \frac{T_0(j\omega_0) + \delta_a}{N_0(j\omega_0) + \delta_b} = \frac{T_0(j\omega_0)}{N_0(j\omega_0) + \delta_{b0}} = \frac{T_0(j\omega_0) + \delta_{a0}}{N_0(j\omega_0)}$$

The example in section 4.5 illustrates how important the second step of the algorithm is. In that example, it improves the phase accuracy while the magnitude accuracy was fair.

Appendix

I. Bessel, Butterworth and Chebyshev transfer functions

This appendix provides analytical expressions for standard low-pass transfer functions, normalized with respect to their asymptotic behavior.

The low-pass Bessel, Butterworth and Chebyshev transfer functions are all-pole transfer functions, well-known from filter applications. Many textbook discussions on these functions normalize them therefore with respect to the –3 dB corner frequency. This normalization is inadequate for specifying the generic pass-band in the synthesis of feedback loops (see section 5.1.3 and 5.3.1).
Band-pass synthesis requires normalization with respect to their *asymptotic* behavior. The all-pole transfer functions H(s) in this appendix, with equal transfer order, have equal transfer for $\omega\to 0$ and for $\omega\to\infty$. Bessel transfer functions are most conveniently calculated from their polynomial coefficients while the use of polynomial roots is to be preferred for the other functions. They are:

$H(s) = 1/N(s)$	$N(s) = \sum_{k=0}^{n} (c_k\cdot(s/\omega_0)^k) = \prod_{k=1}^{n}\left(1-(s/\omega_0)/p_k\right)$	
Bessel	$c_k = c_{k-1}\cdot\frac{(n-k+1)\cdot 2q}{(2n-k+1)\cdot k}$	$\left\{\begin{array}{l} c_0 = 1 \\ q = \sqrt[n]{1\cdot 3\cdot 5\ldots(2n-1)} \end{array}\right\}$
Butterworth	$p_k = \exp(j\pi/2\cdot(2k-1+n)/n)$	
Chebyshev	$p_k = \frac{2\cdot\sinh(\mu/n + j\pi/2\cdot(2k-1+n)/n)}{\lvert 2\cdot\cosh(\mu + j\pi/2\cdot n)\rvert^{1/n}}$	$\left\{\begin{array}{l} \mu = \operatorname{asinh}(1/\varepsilon) \\ \varepsilon_{dB} = 10\cdot\log(1+\varepsilon^2) \end{array}\right\}$

The asymptotic behavior of these functions $N(j\omega)$ is equal for $\omega\to 0$ or $\omega\to\infty$, because their first and last polynomial coefficient are all normalized: $c_0 = c_n = 1$.
In addition to the above definitions, simple analytical expressions exist for the denominator coefficients of Butterworth polynomials. They are:

Butterworth polynomial coefficients (exact)

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
c_0	1	1	1	1	1	1	$c_0 = 1$
c_1	1	$\sqrt{2}$	2	$\sqrt{(4+2\sqrt{2})}$	$1+\sqrt{5}$	$\sqrt{2}+\sqrt{6}$	
c_2		1	2	$2+\sqrt{2}$	$3+\sqrt{5}$	$4+2\cdot\sqrt{3}$	$c_2 = \frac{1}{2}\cdot(c_1)^2$
c_3			1	$\sqrt{(4+2\sqrt{2})}$	$3+\sqrt{5}$	$3\sqrt{2}+2\sqrt{6}$	
c_4				1	$1+\sqrt{5}$	$4+2\cdot\sqrt{3}$	$c_k = c_{n-k}$
c_5					1	$\sqrt{2}+\sqrt{6}$	
c_6						1	

The numerical values of coefficients and poles of all these functions are specified on the next pages.

Bessel polynomial coefficients

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*
c_0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
c_1	1.0000	1.7321	2.4662	3.2011	3.9363	4.6717
c_2		1.0000	2.4329	4.3916	6.8864	9.9202
c_3			1.0000	3.1239	6.7767	12.3583
c_4				1.0000	3.8107	9.6223
c_5					1.0000	4.4952
c_6						1.0000

Butterworth polynomial coefficients

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*
c_0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
c_1	1.0000	1.4142	2.0000	2.6131	3.2361	3.8637
c_2		1.0000	2.0000	3.4142	5.2361	7.4641
c_3			1.0000	2.6131	5.2361	9.1416
c_4				1.0000	3.2361	7.4641
c_5					1.0000	3.8637
c_6						1.0000

Chebyshev polynomial coefficients (1 dB ripple)

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*
c_0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
c_1	1.0000	1.0455	1.9890	1.9522	3.1074	2.8534
c_2		1.0000	1.2525	2.7694	3.4289	5.5888
c_3			1.0000	1.3150	3.9072	4.5796
c_4				1.0000	1.4249	4.7097
c_5					1.0000	1.4497
c_6						1.0000

Chebyshev polynomial coefficients (2 dB ripple)

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*
c_0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
c_1	1.0000	0.8860	2.1541	1.6916	3.4062	2.4928
c_2		1.0000	1.0711	2.7699	3.1162	5.5775
c_3			1.0000	1.0634	4.0834	3.8227
c_4				1.0000	1.1658	4.6943
c_5					1.0000	1.1498
c_6						1.0000

Chebyshev polynomial coefficients (3 dB ripple)

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*
c_0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
c_1	1.0000	0.7665	2.3356	1.4834	3.7419	2.1967
c_2		1.0000	0.9473	2.7790	2.8932	5.5884
c_3			1.0000	0.8967	4.2855	3.2832
c_4				1.0000	0.9998	4.7014
c_5					1.0000	0.9596
c_6						1.0000

Bessel poles *angle in degrees*

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
$\lvert p_1\rvert$	1.0000	1.0000	1.0305	1.0588	1.0825	1.1022	0.0000	30.0000	43.6525	51.6325	56.9366	60.7508	$\angle(-p_1)$
$\lvert p_2\rvert$		1.0000	0.9416	0.9444	0.9598	0.9775		-30.0000	0.0000	16.6697	27.4696	35.1079	$\angle(-p_2)$
$\lvert p_3\rvert$			1.0305	0.9444	0.9264	0.9282			-43.6525	-16.6697	0.0000	11.5411	$\angle(-p_3)$
$\lvert p_4\rvert$				1.0588	0.9598	0.9282				-51.6325	-27.4696	-11.5411	$\angle(-p_4)$
$\lvert p_5\rvert$					1.0825	0.9775					-56.9366	-35.1079	$\angle(-p_5)$
$\lvert p_6\rvert$						1.1022						-60.7508	$\angle(-p_6)$

Butterworth poles *angle in degrees*

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
$\lvert p_1\rvert$	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	0.0000	45.0000	60.0000	67.5000	72.0000	75.0000	$\angle(-p_1)$
$\lvert p_2\rvert$		1.0000	1.0000	1.0000	1.0000	1.0000		-45.0000	0.0000	22.5000	36.0000	45.0000	$\angle(-p_2)$
$\lvert p_3\rvert$			1.0000	1.0000	1.0000	1.0000			-60.0000	-22.5000	0.0000	15.0000	$\angle(-p_3)$
$\lvert p_4\rvert$				1.0000	1.0000	1.0000				-67.5000	-36.0000	-15.0000	$\angle(-p_4)$
$\lvert p_5\rvert$					1.0000	1.0000					-72.0000	-45.0000	$\angle(-p_5)$
$\lvert p_6\rvert$						1.0000						-75.0000	$\angle(-p_6)$

Chebyshev poles (1 dB ripple) *angle in degrees*

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
$\lvert p_1\rvert$	1.0000	1.0000	1.2636	1.3708	1.5121	1.5545	0.0000	58.4846	75.6524	81.9240	84.8372	86.4183	$\angle(-p_1)$
$\lvert p_2\rvert$		1.0000	0.6263	0.7295	0.9966	1.1664		-58.4846	0.0000	50.4085	69.0562	76.8514	$\angle(-p_2)$
$\lvert p_3\rvert$			1.2636	0.7295	0.4403	0.5515			-75.6524	-50.4085	0.0000	48.9176	$\angle(-p_3)$
$\lvert p_4\rvert$				1.3708	0.9966	0.5515				-81.9240	-69.0562	-48.9176	$\angle(-p_4)$
$\lvert p_5\rvert$					1.5121	1.1664					-84.8372	-76.8514	$\angle(-p_5)$
$\lvert p_6\rvert$						1.5545						-86.4183	$\angle(-p_6)$

Chebyshev poles (2 dB ripple) *angle in degrees*

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
$\lvert p_1\rvert$	1.0000	1.0000	1.3665	1.4308	1.6102	1.6116	0.0000	63.7041	78.6996	83.7515	86.0357	87.2606	$\angle(-p_1)$
$\lvert p_2\rvert$		1.0000	0.5355	0.6989	1.0347	1.1971		-63.7041	0.0000	57.4556	73.6398	79.8752	$\angle(-p_2)$
$\lvert p_3\rvert$			1.3665	0.6989	0.3602	0.5183			-78.6996	-57.4556	0.0000	56.3187	$\angle(-p_3)$
$\lvert p_4\rvert$				1.4308	1.0347	0.5183				-83.7515	-73.6398	-56.3187	$\angle(-p_4)$
$\lvert p_5\rvert$					1.6102	1.1971					-86.0357	-79.8752	$\angle(-p_5)$
$\lvert p_6\rvert$						1.6116						-87.2606	$\angle(-p_6)$

Chebyshev poles (3 dB ripple) *angle in degrees*

	n=1	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	*n=1*	*n=2*	*n=3*	*n=4*	*n=5*	*n=6*	
$\lvert p_1\rvert$	1.0000	1.0000	1.4530	1.4651	1.6837	1.6430	0.0000	67.4660	80.6195	84.8580	86.7494	87.7578	$\angle(-p_1)$
$\lvert p_2\rvert$		1.0000	0.4737	0.6825	1.0685	1.2146		-67.4660	0.0000	62.3240	76.4724	81.6867	$\angle(-p_2)$
$\lvert p_3\rvert$			1.4530	0.6825	0.3090	0.5011			-80.6195	-62.3240	0.0000	61.3948	$\angle(-p_3)$
$\lvert p_4\rvert$				1.4651	1.0685	0.5011				-84.8580	-76.4724	-61.3948	$\angle(-p_4)$
$\lvert p_5\rvert$					1.6837	1.2146					-86.7494	-81.6867	$\angle(-p_5)$
$\lvert p_6\rvert$						1.6430						-87.7578	$\angle(-p_6)$

Appendix

J. Algorithm for feedback compensation synthesis

This appendix derives an algorithm that calculates two polynomials required for the compensation synthesis of section 5.3.2. These polynomials represent the modified (profiled) poles and inserted (phantom) zeros of the loop to enable stable feedback operation. The input and output of this compensation algorithm can be denoted as the following function description:

$$[N_x(s),\ T_x(s)] = \mathit{CompSynthesis}(\ g,\ N_0(s),\ T_0(s),\ B_0(s/\omega_0)\ )$$

input/output parameter	**description**
g	the DC loop gain
$N_0(s)$	A denominator polynomial representing the predefined poles of the loop. These are poles of the uncompensated loop, of which the position remains unchanged when the loop is compensated. By definition: $N_0(0)=1$
$T_0(s)$	A numerator polynomial representing the predefined zeros of the loop. These are zeros of the uncompensated loop, of which the position remains unchanged when the loop is compensated. By definition: $T_0(0)=1$
$B_0(s/\omega_0)$	A numerator polynomial of an all-pole low-pass transfer function. This (normalized) low-pass function specifies the generic passband of the effective-aperture. The frequency ω_0 scales the asymptotic bandwidth of this low-pass function. By definition: $B_0(0)=1$ and $B_0(s)\rightarrow s^{n_p}$ for $s\rightarrow j\infty$.
$N_x(s)$	A denominator polynomial representing the modified poles of the loop. These are (profiled) poles of the compensated loop, originating from (profiled) poles of the uncompensated loop. Their new positions facilitate stable operation of the feedback loop. By definition: $N_x(0)=1$
$T_x(s)$	A numerator polynomial representing the inserted (phantom) zeros of the loop. Their positions facilitate stable operation of the feedback loop. By definition: $T_x(0)=1$
compensated loop gain	$\mathring{H}(s) = g \cdot \dfrac{T_0(s)\cdot T_x(s)}{N_0(s)\cdot N_x(s)}$
compensated effective-aperture	$\dfrac{\mathring{H}(s)}{\mathring{H}(s)-1}\cdot\dfrac{1}{T_x(s)} = \dfrac{g}{g-1}\cdot\dfrac{T_0(s)}{T_0(s)}\cdot\dfrac{1}{B_0(s/\omega_0)}$
$T_0(s)$	The polynomial $T_0(s)$ is nearly equal to $T_0(s)$, with the exception of roots in the right complex half-plane (RHP). $T_0(s)$ represents the combination of all LHP-poles, and mirrored RHP poles of $T_0(s)$. The transfer function $T_0(s)/T_0(s)$ is a pseudo delay transfer function with unity magnitude response for all frequencies

The purpose of this appendix is to find poles $N_x(s)$ and zeros $T_s(s)$ facilitating the effective-aperture to meet a predefined low-pass transfer function. The roots of B(s) provide a generic form of this transfer function, however, it requires adequate scaling to

match it with the asymptotic properties of the loopgain. Our compensation algorithm is focused on facilitating the following transfer of the effective-aperture:

$$\underbrace{A_{sa}(s)}_{\text{system aperture}} = \underbrace{\frac{g}{g-1}}_{\text{nearly unity gain}} \cdot \underbrace{\frac{T_0(s)}{T_0(s)}}_{\text{pseudo delay}} \cdot \underbrace{\frac{1}{B_0(s/\omega_0)}}_{\text{pass-band of the loop}} = \frac{1}{1-1/\mathring{H}(s)} \cdot \frac{1}{T_x(s)}$$

magnitude scaling *delay scaling* *bandwidth scaling*

What loopgain $\mathring{H}(s)$ is required to facilitate a effective-aperture with this transfer function? To find an answer, the polynomials $N_x(s)$ and $T_x(s)$ will be reconstructed from this relation. Reordering results in:

$$\begin{aligned}
&(1-1/\mathring{H}(s)) \cdot T_x(s) && = (1-1/g) \cdot T_0(s)/T_0(s) \cdot B_0(s/\omega_0)\\
&(1-1/\mathring{H}(s)) \cdot T_x(s) \cdot T_0(s) && = (1-1/g) \cdot T_0(s) \cdot B_0(s/\omega_0)\\
&T_x(s) \cdot T_0(s) \quad -1/g \cdot N_x(s) \cdot N_0(s) && = (1-1/g) \cdot T_0(s) \cdot B_0(s/\omega_0)\\
&N_x(s) \cdot N_0(s) \quad -g \cdot T_x(s) \cdot T_0(s) && = (1-g) \cdot T_0(s) \cdot B_0(s/\omega_0)\\
&N_0(s) \cdot (N_x(s)-1) \quad -g \cdot T_0(s) \cdot (T_x(s)-1) && = (1-g) \cdot T_0(s) \cdot B_0(s/\omega_0) - N_0(s) + g \cdot T_0(s)
\end{aligned}$$

To simplify this equation, we define the polynomials $Q_x(s)$, $R_x(s)$ and $\Upsilon_0(s)$ as temporarily substitution quantities.

$$\begin{aligned}
Q_x(s) &\stackrel{def}{=} (1/s) \cdot (N_x(s)-1)\\
R_x(s) &\stackrel{def}{=} (1/s) \cdot (T_x(s)-1) \cdot (-g)\\
\Upsilon_0(s) &\stackrel{def}{=} (1/s) \cdot ((1-g) \cdot T_0(s) \cdot B_0(s/\omega_0) - N_0(s) + g \cdot T_0(s))
\end{aligned}$$

$$N_0(s) \cdot Q_x(s) + T_0(s) \cdot R_x(s) = \Upsilon_0(s)$$

This substitution has reduced the complexity of the equation to a standard format. The unknown polynomials $Q_x(s)$ and $R_x(s)$ result from a weighed polynomial division (see appendix *G*) of $\Upsilon_0(s)$ by $N_0(s)$ using $T_0(s)$ as weighing polynomial.

The formal description of the compensation algorithm is:

$[N_x(s), T_x(s)] = \textit{CompSynthesis}(\, g, N_0(s), T_0(s), B_0(s/\omega_0)\,)$

$T_0(s) = \text{LHP}(T_0)$ =copy all LHP roots and mirror all RHP roots

$\Upsilon_0(s) = (1/s) \cdot ((1-g) \cdot T_0(s) \cdot B_0(s/\omega_0) - N_0(s) + g \cdot T_0(s))$

$[Q_x, R_x] = \text{polydiv}(\Upsilon_0, N_0, T_0)$ = weighed polynomial division with T_0 (see appendix *G*)

$N_x(s) = 1 + s \cdot Q_x(s)$

$T_x(s) = 1 - s \cdot R_x(s)/g$

Appendix

K. Definitions of spectra and integral transformations

This appendix provides mathematical relations between signal representations in the time domain and the frequency domain; it is the theoretical base for section 7.1.
Spectral methods are powerful method for analyzing harmonic and random signals. The concept of *spectrum* is not uniquely defined. They are usually a modification of the standard Fourier-transform, since the Fourier-transform of (pure) harmonic and random signals does not exist. This is because the Fourier integral of infinite long time intervals is not convergent.
We note that most textbook discussions and publications use the letter S for a quantity named *power spectrum, spectral power density, spectral density, spectral intensity, noise power* or *RMS noise* however some of them refer to a ***single*** sided spectrum while others meant a ***double*** sided spectrum. This indicates the relevance of accurate definitions of what spectrum is used.

averaging			
average, around t=0	$\langle\, q(t) \,\rangle_T$	$\stackrel{\text{def}}{=} \frac{1}{T}\cdot\int_{-½T}^{+½T} q(t)\cdot dt$	T = system resolution or system integration time
average, around t=t₀	$\langle\, q(t) \,\rangle_{T,\,t_0}$	$\stackrel{\text{def}}{=} \frac{1}{T}\cdot\int_{t_0-½T}^{t_0+½T} q(t)\cdot dt$	
stationairy value	U_0	$\stackrel{\text{def}}{=} \langle\, U(t) \,\rangle_\infty$	
average intensity	$(U_{rms})^2$	$\stackrel{\text{def}}{=} \langle\, \lvert U(t)\rvert^2 \,\rangle_\infty = \langle\, U(t)\cdot U^*(t) \,\rangle_\infty$	

($u^2(t)$ = instant intensity)
Note that $U^*(t)$ *is the complex conjugate of* $U(t)$ *and that* $U'(t)$ *is the conjugated transpose*[1] *of a signal.*

transformation to the frequency domain			
$\mathbf{j}_d\{U(t);T\}$	$\stackrel{\text{def}}{=}$	$\frac{1}{T}\cdot\int_{-½T}^{+½T} U(t)\cdot\exp(-j\cdot 2\pi nt/T)\cdot dt$	*(discrete spectrum)*
$\mathbf{j}_n\{U(t);T\}$	$\stackrel{\text{def}}{=}$	$\frac{1}{\sqrt{½T}}\cdot\int_{-½T}^{+½T} U(t)\cdot\exp(-j\cdot 2\pi ft)\cdot dt$	*(noise spectrum)*
$\mathbf{j}_c\{U(t)\}$	$\stackrel{\text{def}}{=}$	$\int_{-\infty}^{+\infty} U(t)\cdot\exp(-j\cdot 2\pi ft)\cdot dt$	*(continuous spectrum)*
$\boldsymbol{D}\{U(t)\}$	$\stackrel{\text{def}}{=}$	$\int_{-\infty}^{+\infty} \langle\, U(t)\cdot U^*(t+\tau) \,\rangle_\infty\cdot\exp(-j\cdot 2\pi f\tau)\cdot d\tau$	*(double sided intensity spec.)*
$\boldsymbol{S}\{U(t)\}$	$\stackrel{\text{def}}{=}$	$2\cdot\int_{-\infty}^{+\infty} \langle\, U(t)\cdot U^*(t+\tau) \,\rangle_\infty\cdot\exp(-j\cdot 2\pi f\tau)\cdot d\tau$	*(single sided intensity spec.)*

[1] Note that U(t) represents a scalar quantity, and not a vector or matrix of scalar quantities. For scalar quanties holds that $U'(t) \equiv U^*(t)$, where $U^*(t)$ represents the complex conjugate of U(t). For matrix quantities $\mathbf{U}'(t) \equiv \mathbf{U}^{*T}(t)$.

When the interpretation is unambiguous, the following shortcuts may be used:

$\mathbf{j}^{¥}_{d}\{U(t);T\}$	$\rightarrow \ddot{\ddot{Q}}_u(n)$	$\rightarrow \ddot{\ddot{U}}_n$	$\ddot{\ddot{U}}\ \ddot{\ddot{V}}\ \ddot{\ddot{I}}\ \ddot{\ddot{J}}$
$\mathbf{j}_{n}\{U(t);T\}$	$\rightarrow N_u(f)$	$\rightarrow U(f)$	U V I J
$\mathbf{j}_{c}\{U(t)\}$	$\rightarrow F_u(f)$	$\rightarrow U(f)$	U V I J
$\boldsymbol{D}\{U(t)\}$	$\rightarrow D_u(f)$		
$\boldsymbol{S}\{U(t)\}$	$\rightarrow S_u(f)$		

The *continuous* Fourier spectrum is only defined for functions that vanish to zero when the time goes to infinity. This Fourier transform is therefore *undefined* for signals such as noise and for periodical signals.

The Fourier *noise* spectrum uses a finite interval T, and is therefore wider applicable. The magnitude of the spectrum increases with the length of the measurement interval, however this effect is compensated by a scaling factor ($1/\sqrt{½T}$) in the special case that random noise is measured. It facilitates that the magnitude of the spectrum, *averaged* over a 'small' frequency band, converges to a finite value when T goes to infinity.

The *discrete* Fourier spectrum is also based on a finite interval, however the associated scaling factor (1/T) is optimized for periodical functions. The magnitude of its spectral components, associated with interval T_1, equals to that of T_2 when $f(t)=f(t+T_1)=f(t+T_2)$. A Spectrum analyzer indicates a spectrum that is close to the Fourier noise spectrum for random signals, and a spectrum that is close to the discrete Fourier spectrum for periodical signals.

reverse transformation to the time domain

$U(t)$	$=$	$\sum_{n=-\infty}^{+\infty} \ddot{\ddot{Q}}_u(n)\cdot\exp(+j\cdot 2\pi nt/T)$	*periodic function f(t)=f(t+T)*
$U(t)$	$=$	$\sqrt{½T}\cdot\int_{-\infty}^{+\infty} N_u(f)\cdot\exp(+j\cdot 2\pi ft)\cdot df$	*for (|t|<½T)*
$U(t)$	$=$	$\int_{-\infty}^{+\infty} F_u(f)\cdot\exp(+j\cdot 2\pi ft)\cdot df$	

time-domain correlation

auto correlation:	$\boldsymbol{R}\{U(t)\}$	$\stackrel{def}{=} R_{uu}(\tau)$	$\stackrel{def}{=} \langle U(t)\cdot U^*(t+\tau)\rangle_\infty$
cross correlation:	$\boldsymbol{R}\{U(t);V(t)\}$	$\stackrel{def}{=} R_{uv}(\tau)$	$\stackrel{def}{=} \langle U(t)\cdot V^*(t+\tau)\rangle_\infty$

$$\langle U(t)\cdot V^*(t+\tau)\rangle_{\infty,t_0} = \langle U(t)\cdot V^*(t+\tau)\rangle_{\infty,t_0+\Delta t}$$
$$\langle U(t)\cdot V^*(t+\tau)\rangle_{T,t_0} = \langle U(t-\tau)\cdot V^*(t)\rangle_{T,t_0+\tau}$$
$$\langle U(t)\cdot V^*(t+\tau)\rangle_{\infty} = \langle U(t-\tau)\cdot V^*(t)\rangle_{\infty}$$

spectral correlation

$$S_u(f) = 2\cdot\int_{-\infty}^{+\infty} R_u(\tau)\cdot\exp(-j\cdot 2\pi f\tau)\cdot d\tau = 4\cdot\int_{0}^{+\infty} R_u(\tau)\cdot\cos(2\pi f\tau)\cdot d\tau$$

$$D_u(f) = \int_{-\infty}^{+\infty} R_u(\tau)\cdot\exp(-j\cdot 2\pi f\tau)\cdot d\tau = 2\cdot\int_{0}^{+\infty} R_u(\tau)\cdot\cos(2\pi f\tau)\cdot d\tau$$

$$R_u(t) = \int_{-\infty}^{+\infty} D_u(f)\cdot\exp(+j\cdot 2\pi f t)\cdot df = \int_{0}^{+\infty} S_u(f)\cdot\cos(2\pi f t)\cdot df$$

$$\mathbf{j}_c\{\langle U(t)\cdot V^*(t+\tau)\rangle_\infty\} = \mathbf{j}_c\{\langle V(t)\cdot U^*(t+\tau)\rangle_\infty\}^*$$

$$\boldsymbol{S}\{U(t)\} = 2\cdot\mathbf{j}_c\{\langle U(t)\cdot U^*(t+\tau)\rangle_\infty\}$$

$$\boldsymbol{S}\{U(t)+V(t)\} = \boldsymbol{S}\{U(t)\} + \boldsymbol{S}\{V(t)\} + 4\cdot Real\{\mathbf{j}_c\{\langle U(t)\cdot V^*(t+\tau)\rangle_\infty\}\}$$

$$\underline{\boldsymbol{S}}\{U(t);V(t)\} \stackrel{def}{=} \begin{pmatrix} S_{uu}(f) & S_{uv}(f) \\ S_{vu}(f) & S_{vv}(f) \end{pmatrix} \stackrel{def}{=} \mathbf{j}_c\begin{Bmatrix} \langle U(t)\cdot U^*(t+\tau)\rangle_\infty & \langle U(t)\cdot V^*(t+\tau)\rangle_\infty \\ \langle V(t)\cdot U^*(t+\tau)\rangle_\infty & \langle V(t)\cdot V^*(t+\tau)\rangle_\infty \end{Bmatrix}$$

$$\underline{\boldsymbol{S}}\{J(t);V(t)\} \stackrel{def}{=} \begin{bmatrix} S_{jj}(f) & S_{jv}(f) \\ S_{vj}(f) & S_{vv}(f) \end{bmatrix} \stackrel{def}{=} \mathbf{j}_c\begin{Bmatrix} \langle J_d(t)\cdot J_d^*(t+\tau)\rangle_\infty & \langle J_d(t)\cdot V_d^*(t+\tau)\rangle_\infty \\ \langle V_d(t)\cdot J_d^*(t+\tau)\rangle_\infty & \langle V_d(t)\cdot V_d^*(t+\tau)\rangle_\infty \end{Bmatrix}$$

$S_{uu}(f)$ = *self-spectral intensity*
$S_{uv}(f)$ = *cross-spectral intensity*

intensity relations (Parceval identities)

$$\int_{-\infty}^{+\infty} F_{u1}(f)\cdot F_{u2}^*(f)\cdot df = \int_{-\infty}^{+\infty} U_1(t)\cdot U_2^*(t)\cdot dt$$

$$\int_{-\infty}^{+\infty} N_{u1}(f)\cdot N_{u2}^*(f)\cdot df = \langle U_1(t)\cdot U_2^*(t)\rangle_T$$

$$\sum_{n=-\infty}^{+\infty} \breve{\breve{Q}}_{u1}(n)\cdot\breve{\breve{Q}}_{u2}(n) = \langle U_1(t)\cdot U_2^*(t)\rangle_T$$

$$\int_{-\infty}^{+\infty} |F_u(f)|^2\cdot df = \int_{-\infty}^{+\infty} |U(t)|^2\cdot dt$$

$$\int_{-\infty}^{+\infty} |N_u(f)|^2\cdot df = \langle\, |U(t)|^2\,\rangle_T$$

$$\sum_{n=-\infty}^{+\infty} |\breve{\breve{Q}}_u(n)|^2 = \langle U_1(t)\cdot U_2^*(t)\rangle_T$$

$$\langle |U(t)|^2\rangle_\infty = \langle U(t)\cdot U^*(t)\rangle_\infty = R_u(0) = \int_{0}^{+\infty} S_u(f)\cdot df = \int_{-\infty}^{+\infty} D_u(f)\cdot df$$

Wiener-Khintchine theorem

This approximation holds for 'sufficient' small resolution bandwidth Δf, and 'sufficient' large measurement interval T. It relates the average spectral magnitude of a noise spectrum to an intensity spectrum:

$$\left\langle |N_u(f)|^2 \right\rangle_{\Delta f,f} \approx S_u(f) = 2 \cdot D_u(f)$$

miscellaneous

$\langle U(t) \rangle_T = N_{u,T}(0) / \sqrt{T} = \overset{\infty}{Q}_{u,T}(0)$

$\left\langle |U(t)|^2 \right\rangle_\infty = R_u(0)$

$R_u(+t) = R_u^*(-t) =$ *real*
$S_u(+f) = S_u(-f) =$ *real*
$\overset{\infty}{Q}_u(n) = F_u(f_n) \cdot \Delta f$ *when* $U(t)=0$ *for* $(|t|>½T)$ $(\Delta f=1/T; \quad f_n=n\cdot\Delta f)$

convolution product (smoothing operator)

convolution product

$$(U \otimes V)(\tau) \overset{def}{=} U(t) \otimes V(t) \overset{def}{=} \int_{-\infty}^{+\infty} U(t)\cdot U(t-\tau)\cdot dt \approx T \cdot \left\langle U(t)\cdot V(t-\tau) \right\rangle_T$$

commutation:	$U \otimes V$	$= V \otimes U$
association:	$U \otimes (V \otimes W)$	$= (U \otimes V) \otimes W$
distribution:	$U \otimes (V+W)$	$= (U \otimes V) + (U \otimes W)$

Appendix

L. Algorithm for extraction of equivalent input noise

This appendix describes an algorithm for reconstructing the equivalent excess noise, at the input of an amplifier, relative to well-defined reference planes and source admittances. It requires the detection of the amplifier output noise, while additional noise is supplied to the input using a variable (white) noise source. The source admittance of interest is simulated using an admittance tuner.
At least two output noise levels are required for the extraction. For a description of the application of this algorithm, see section 8.5.3 and Mahmoudi [834]. The basic measurement setup and the various mathematical transformation steps are illustrated in figure 10.1.

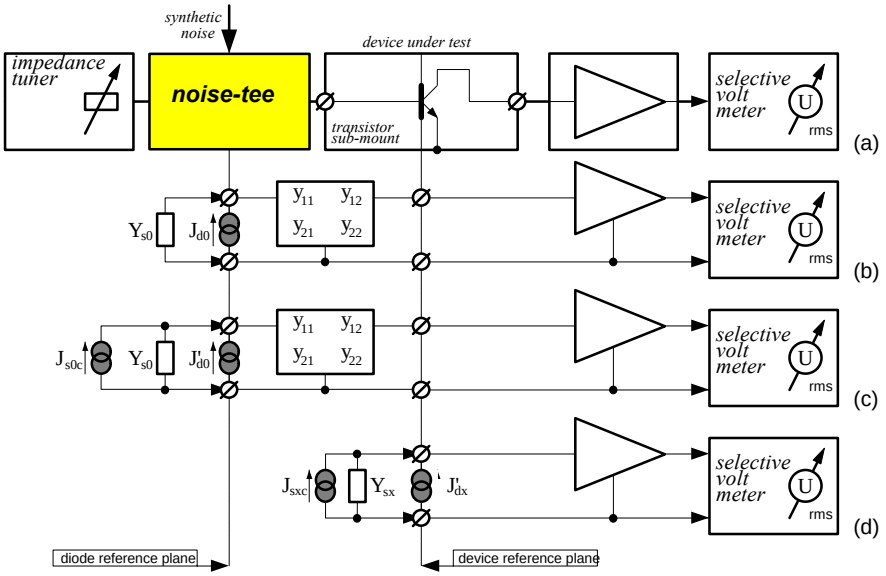


Fig 10.1 *Basic measurement setup (a) for the measurement of device noise J_{d0} observed from the diode reference plane for specified admittance Y_{s0}. In (b,c,d) the various steps are illustrated to transform this noise (including thermal noise effects) to the device excess-noise J'_{dx} observed from the device reference plane (excluding thermal noise effects J_{sxc}).*

- Figure 10.1a shows a noise-tee. One port is connected with a device under test and the other port is connected with an admittance tuner, to adjust a source admittance of interest. The measurement goal is to extract the equivalent excess-noise, observed from the device reference plane, associated with this source admittance.
- Figure 10.1b shows the measurement of device noise J_{d0}. The noise-tee is illuminated with two or more calibrated synthetic noise levels, to improve overall measurement accuracy. The admittance tuner is adjusted to a specific admittance Y_{s0} of interest. For each reference level S_{rx}, the associated output 'power' P_{rx} is extracted for arbitrary but

sufficient small measurement bandwidth. This 'power' is defined as the square of the detected rms-voltage.
These measurements facilitate the extraction of the equivalent device noise 'observed' from the diode reference plane. The total equivalent noise, 'observed' from this reference plane includes thermal noise effects of tuner admittance and noise-tee losses. The intensity spectrum S_{Jd0} of the equivalent device noise current J_{d0} is extracted as follows, using the algorithm of section 8.4.3:

$$\begin{bmatrix} G\cdot S_{Jd0} \\ \\ G\cdot S_0 \end{bmatrix} = \begin{bmatrix} 1/P_{r1} & (S_{r1}/S_0)/P_{r1} \\ 1/P_{r2} & (S_{r2}/S_0)/P_{r2} \\ \dots & \dots \\ 1/P_{rn} & (S_{rn}/S_0)/P_{rn} \end{bmatrix} \backslash \begin{bmatrix} 1 \\ 1 \\ . \\ 1 \end{bmatrix}$$

$\begin{cases} G = \text{the virtual power 'gain' of the setup} \\ S_0 = \text{an arbitrary scaling factor} \\ S^0_{rx} = \text{the reference noise levels} \\ P_{rx} = \text{the detected output power} \end{cases}$

• Figure 10.1c illustrates the extraction of the device *excess-noise* J'_{d0} from J_{d0} and the tuner admittance. In section 7.2.5 we discussed the relation of thermal noise of a passive network with the tuner admittance observed from the diode reference plane. Since this reference plane is not accessible for direct admittance measurement, we reconstruct its value from Y_{sx} using the de-embedding equations of section 2.3.1 and section 8.3.4. Assume, for the time being, that the embedded two-port parameters $[y_{11},y_{21},y_{12},y_{22}]$ and the tuner admittance Y_{sx} are known from other measurements. Then the spectral intensity S'_{Jd0} of the excess-noise becomes:

$$S'_{Jd0} = S_{Jd0} - 4kT\cdot real(Y_{s0}) \quad = \quad S_{Jd0} - 4kT\cdot real\left\{\frac{\Delta_y - Y_{sx}\cdot y_{11}}{Y_{sx} - y_{22}}\right\}$$

• Figure 10.1d shows the transformation of the device *excess-noise*, observed from the diode reference plane (J'_{d0}) to the device reference plane (J'_{dx}). Using the de-embedding equations of section 8.3.4 and section 2.3.1 to relate J'_{dx} with J'_{d0}, then we obtain for their intensity spectra:

$$J'_{dx} = \left(\frac{Y_{sx} - y_{22}}{y_{12}}\right)\cdot J'_{d0} \qquad \Rightarrow \qquad S'_{Jdx} = \left|\frac{Y_{sx} - y_{22}}{y_{12}}\right|^2 \cdot S'_{Jd0}$$

$$S'_{Jdx} = \left|\frac{Y_{sx} - y_{22}}{y_{12}}\right|^2 \cdot \left(S_{Jd0} - 4kT\cdot real\left\{\frac{\Delta_y - Y_{sx}\cdot y_{11}}{Y_{sx} - y_{22}}\right\} \right)$$

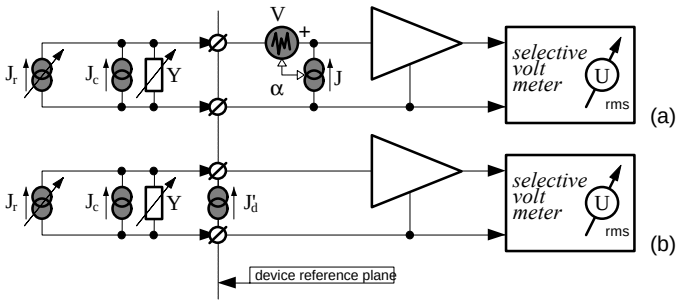
Spectral intensity of device excess noise current.

The thermal noise J_{s0c} is transformed in a similar way. Since the losses in noise-tee and transistor fixture introduce additional thermal noise effects, it is more convenient to combine all these thermal noise effects. This yields an overall extraction of the spectral intensity of the total thermal noise current J_{sxc}, using $S_{Jsxc}=real(Y_{sx})$.
The admittance Y_{sx} is obtained from measurements, using a duplicate of the transistor fixture. Cutting this duplicate fixture into two pieces, gives full access to the (duplicated) device reference plane. It enables admittance measurements using well-known network analyzer techniques.
The embedded two-port parameters $[y_{11},y_{21},y_{12},y_{22}]$ are obtained from full two-port measurements on both ports of the noise-tee, followed by mathematical halving of this two-port using the algorithms of section 8.3.5.

Appendix

M. Noise parameter extraction using hot and cold sources

This appendix describes an algorithm for reconstructing the two-port noise parameters of devices such as transistors. The algorithm requires the measured output noise levels when the device under test (DUT) is connected to a noise source with known source impedance and noise levels, and the DUT input admittance in known. Noise level and noise source admittance are varied to vary the detected output noise levels.
This method is a generalized variant of the methods of Adamian and Uhlir [825,830] and Davidson et al. [831], as described in section 8.5.1. Figure 10.2 shows the measurement principle, as described in section 8.5.3, and a linear relation for device excess-noise intensity S_i using $[x_1,x_2,x_3,x_4]$ as noise parameters
.



$$S_i = x_1 + |Y|^2 \cdot x_2 + re(Y) \cdot x_3 + im(Y) \cdot x_4$$

Fig 10.2 *The device excess-noise current J'_{d0}, with spectral intensity S_i, is obtained for arbitrary source admittance values Y from a full noise representation of the input port of the device under test (a). Current J_c is the thermal noise of the tunable source admittance Y and J_r is the tunable source excess noise. The measurement of the noise parameters (c) is performed with a noise source, tunable in reference noise level and in reference source admittance.*

The source has admittance Y, (cold) thermal noise current J_c. and an additional reference noise J_r. The noise parameters are extracted from several measurements with these reference noise levels and admittances and from the DUT[2] input admittance. Since all reference admittances are first to be measured, additional measurement of the input admittance Y_{in} will hardly increase the overall measurement effort. The sensitivity of modern network analyzers enables accurate admittance measurements with low stimulus level to avoid overloading the device under test.
Let P be the detected output noise power (the square of the rms-voltage), when the DUT is connected to a noise source with reference admittance Y. Let $S_{i,tot}$ (or $S_{u,tot}$) be the *total* noise intensity of the equivalent current (or voltage) that is concentrated at the input of the device under test when it is connected to that source. This is the addition of the known (1) thermal noise of the source admittance, the known (2) excess-noise intensity S_r of the noise source, and the unknown (3) excess-noise intensity S_i of the

[2] DUT = Device under test

DUT. Furthermore, let G be the virtual power gain of the measurement setup, as was defined in section 8.4.1.
Since four unknown noise parameters $[x_1,x_2,x_3,x_4]$ and an additional gain parameter G are to be extracted, at least five noise measurements are required under different circumstances. The detected output power for each noise measurement equals:

$$P = G \cdot \{S_{u,tot}\}$$
$$P = G / |Y+Y_{in}|^2 \cdot \{S_{i,tot}\}$$
$$P = G / |Y+Y_{in}|^2 \cdot \{S_i + S_r + 4\cdot kT\cdot re(Y)\}$$
$$P = G / |Y+Y_{in}|^2 \cdot \{x_1 + |Y|^2\cdot x_2 + re(Y)\cdot x_3 + im(Y)\cdot x_4 + S_r + 4\cdot kT\cdot re(Y)\}$$

$$let \quad h \quad \stackrel{def}{=} 1/|Y+Y_{in}|^2$$
$$let \quad H \quad \stackrel{def}{=} 1/|Y+Y_{in}|^2/P$$

$$P = G \cdot h \cdot \{x_1 + |Y|^2\cdot x_2 + re(Y)\cdot x_3 + im(Y)\cdot x_4 + S_r + 4\cdot kT\cdot re(Y)\}$$

The algorithm that we propose extracts the five unknown variables from a set with five or more of these simultaneous equations. Note that there is no serious restriction to the noise source, as long as more than one 'temperature' is used, and four or more different admittance values. One 'hot' measurement and four (or more) passive admittances are adequate. Some multi level measurements (several different 'temperatures' at equal source admittance) are also permitted. Using many levels and admittance values will improve the overall measurement accuracy.

Assume that the relative accuracy of the admittance measurements and of the reference noise levels are superior to the detection errors of the noise power. Then, optimization of the relative error δ in the detected noise power $P\cdot(1+\delta)$ is equivalent to optimization of the relative accuracy of $S_{i,tot}$. We assume that this minimization goal provides the most plausible solution for $[x_1,x_2,x_3,x_4,G]$ when the source admittance that will be used in the application is not specified.
To find this solution, we will minimize the *relative* error δ_i. This relative minimization makes the average accuracy of the predicted values S_i insensitive to the admittance Y of interest. Let's consider the following set of simultaneous equations, and solve it with the left-hand matrix division of appendix *B*:

$$\begin{bmatrix} P_1(1+\delta_1) \\ P_2(1+\delta_2) \\ \dots \\ \dots \\ \dots \\ P_n(1+\delta_n) \end{bmatrix} = \begin{bmatrix} h_1 & h_1\cdot|Y_1|^2 & h_1\cdot re(Y_1) & h_1\cdot im(Y_1) & h_1\cdot(S_{r1}+4kT\cdot re(Y_1)) \\ h_2 & h_2\cdot|Y_2|^2 & h_2\cdot re(Y_2) & h_2\cdot im(Y_2) & h_2\cdot(S_{r2}+4kT\cdot re(Y_2)) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ h_n & h_n\cdot|Y_n|^2 & h_n\cdot re(Y_n) & h_n\cdot im(Y_n) & h_n\cdot(S_{rn}+4kT\cdot re(Y_n)) \end{bmatrix} \cdot \begin{bmatrix} G\cdot x_1 \\ G\cdot x_2 \\ G\cdot x_3 \\ G\cdot x_4 \\ G \end{bmatrix}$$

$$\begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \\ \dots \\ \dots \\ \dots \\ 1+\delta_n \end{bmatrix} = \begin{bmatrix} H_1 & H_1\cdot|Y_1|^2 & H_1\cdot re(Y_1) & H_1\cdot im(Y_1) & H_1\cdot(S_{r1}+4kT\cdot re(Y_1)) \\ H_2 & H_2\cdot|Y_2|^2 & H_2\cdot re(Y_2) & H_2\cdot im(Y_2) & H_2\cdot(S_{r2}+4kT\cdot re(Y_2)) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ H_n & H_n\cdot|Y_n|^2 & H_n\cdot re(Y_n) & H_n\cdot im(Y_n) & H_n\cdot(S_{rn}+4kT\cdot re(Y_n)) \end{bmatrix} \cdot \begin{bmatrix} G\cdot x_1 \\ G\cdot x_2 \\ G\cdot x_3 \\ G\cdot x_4 \\ G \end{bmatrix}$$

$$\begin{bmatrix} G\cdot x_1 \\ G\cdot x_2 \\ G\cdot x_3 \\ G\cdot x_4 \\ G \end{bmatrix} = \begin{bmatrix} H_1 & H_1\cdot|Y_1|^2 & H_1\cdot re(Y_1) & H_1\cdot im(Y_1) & H_1\cdot(S_{r1}+4kT\cdot re(Y_1)) \\ H_2 & H_2\cdot|Y_2|^2 & H_2\cdot re(Y_2) & H_2\cdot im(Y_2) & H_2\cdot(S_{r2}+4kT\cdot re(Y_2)) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ H_n & H_n\cdot|Y_n|^2 & H_n\cdot re(Y_n) & H_n\cdot im(Y_n) & H_n\cdot(S_{rn}+4kT\cdot re(Y_n)) \end{bmatrix} \backslash \begin{bmatrix} 1 \\ 1 \\ \dots \\ \dots \\ \dots \\ 1 \end{bmatrix}$$

$$H_x \stackrel{def}{=} 1/\,|Y_x+Y_{in}|^2/P_x$$

The above expression provides the noise parameters $[x_1,x_2,x_3,x_4]$, without extracting the excess-noise for each source admittance of use.
When all hot sources are matched and all cold sources are arbitrary admittances, then a reference plane transformation is avoided (see sections 8.5.3 and 8.3.4). This can be implemented simply with a noise-tee or a conventional 50Ω noise source. Nevertheless, our generalized algorithm is wider applicable.
When the noise parameters are intended for a wide range of source impedances, choose admittance values that are spread out over the unity reflection circle. For instance: a *load* ($Z=Z_0=50\Omega$), an *open* ($Z=\infty$), a *short* $Z=0$, a *capacitor* ($Z=-j\cdot Z_0$) and an *inductor* ($Z=+j\cdot Z_0$). Otherwise, chose several of them close to the source admittance of the application.

Appendix

N. Noise parameter extraction using paired hot and cold sources

This appendix describes another algorithm for reconstructing the two-port noise parameters of devices such as transistors, similar to the algorithm in appendix *M*. The measurement setup is equal to figure 10.2 and requires a noise source with variable noise level and variable source admittance.
The major difference is that the algorithm requires no information on the input impedance of the device under test. This is an advantage because, the input admittance Y_{in} of the DUT is not always known or available for measurement.
The draw-back of this method, compared to the method of appendix *M* is that measurements must be performed in *pairs*. Each pair is measured with equal source admittance, with two or more noise levels. This is a classic approach (see section 8.5.1, method 2) improved by Lane [823].
At least seven noise measurements are required for defining a set simultaneous equations that has a unique solution for all seven unknown $[x_1,x_2,x_3,x_4, G, re\{Y_{in}\}, im\{Y_{in}\}]$. Unfortunately, this results in a non-linear equation set and will probably require iterative algorithms to find that solution.
When at least four *pairs* of noise measurements are performed, then the equation set can be transformed into a linear form.

The methods of section 8.5.2 are adequate for finding a solution for each pair, associated with admittance Y. When more levels are used in a pair, and when more pairs are used in the overall measurement, then the overall accuracy will improve.
Assume that each pair provides a value for the reconstructed intensity S_i of the device excess-noise current. Furthermore assume that each reconstruction is performed with some relative error δ. As a result, each pair provides the constants for the linear equation:

$$S_i \approx (1+\delta_i)\cdot S_i = x_1 + |Y|^2\cdot x_2 + re(Y)\cdot x_3 + im(Y)\cdot x_4$$

A set of four or more of these simultaneous equations is adequate to find a solution for the noise parameters $[x_1,x_2,x_3,x_4]$. Let's consider the following set of simultaneous equations, and solve it with the left-hand matrix division of appendix *B*:

$$\begin{bmatrix} 1 & |Y_1|^2 & re(Y_1) & im(Y_1) \\ 1 & |Y_2|^2 & re(Y_2) & im(Y_2) \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1 & |Y_n|^2 & re(Y_n) & im(Y_n) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} S_{i1}\cdot(1+\delta_1) \\ S_{i2}\cdot(1+\delta_2) \\ \dots \\ \dots \\ S_{in}\cdot(1+\delta_n) \end{bmatrix}$$

$$\begin{bmatrix} 1/S_{i1} & |Y_1|^2/S_{i1} & re(Y_1)/S_{i1} & im(Y_1)/S_{i1} \\ 1/S_{i2} & |Y_2|^2/S_{i2} & re(Y_2)/S_{i2} & im(Y_2)/S_{i2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 1/S_{in} & |Y_n|^2/S_{in} & re(Y_n)/S_{in} & im(Y_n)/S_{in} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1+\delta_1 \\ 1+\delta_2 \\ \dots \\ \dots \\ 1+\delta_n \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/S_{i1} & |Y_1|^2/S_{i1} & re(Y_1)/S_{i1} & im(Y_1)/S_{i1} \\ 1/S_{i2} & |Y_2|^2/S_{i2} & re(Y_2)/S_{i2} & im(Y_2)/S_{i2} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ 1/S_{in} & |Y_n|^2/S_{in} & re(Y_n)/S_{in} & im(Y_n)/S_{in} \end{bmatrix} \backslash \begin{bmatrix} 1 \\ 1 \\ .. \\ .. \\ 1 \end{bmatrix}$$

The left-hand matrix division minimized the error δ_i in a least-squares sense, which is the *relative* error of the reconstructed input excess-noise current. This relative minimization makes the average accuracy of the predicted values S_i insensitive to the admittance Y of interest.

Appendix

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Appendix

P. Bibliography of the author

Rob F.M. van den Brink, was born in Oegstgeest, the Netherlands, in 1955. His route to a university degree was not the most obvious one (MULO, HAVO, HTS). In 1984, he graduated from the Technical University Delft in the Netherlands, with a degree in electrical engineering. His graduation project concerned the development of feeds for microwave antennas.

He joined the same university to work on microwave remote sensing experiments, but changed in 1985 to join PTT Research in the Netherlands. Since then, he worked on fiber-optical transmission systems. He developed wideband optical receivers and transmitters up to a few GHz, and various new measurement and software design tools for opto-electronic circuits. His group designed various analog circuits for coherent optical transmission systems, for internal projects, for an ESPRIT 2054 (UCOL) project supported by the EEC, and later for a RACE 2024 (BAF) project.

These activities formed the technical base for this thesis study. In the period from January 1992 to April 1994, his work was focused on developing the theories and algorithms for this Ph.D.-thesis and on writing this text and various publications.
Two patent applications (pending) resulted from this thesis work.

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[123] **MATLAB**: an interactive mathematical computing encironment for numerical computation and visualization. Courtesy of The Mathworks Inc. Natick Mass. USA, FAX: (508)653-2997

[124] **TOUCHSTONE**: a linear circuit simulator with support of microwave circuit elements, s-parameters, measured data, and optimization. Courtesy of EEsof Inc., Westlake Village, USA.

[125] **LIBRA**: A non-linear circuit simulator using harmonic balance techniques, compatible with Touchstone. Courtesy of EEsof Inc., Westlake Village, USA.

[126] **MDS**: A circuit simulation environment (linear and non-linear), support of microwave circuit elements, s-parameters, measured data, and optimization. Courtesy of Hewlett Packard Company, Santa Clara, USA
[127] **SPICE**: A public domain circuit simulator (linear and non-linear), developed during the mid-1970s. from the University of California, Berkeley, USA.
[128] **PSPICE**: One of the many commercial derivatives of SPICE. Courtesy of MicroSim Corporation, Irvine, California, USA.

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Appendix

R. Index

D

E

F

G

H

I

O

P

R

S

T

V

W

Y

Z